DFT AND BELIEF REVISION

EDUARDO FERMÉ
Departamento de Matemática e Engenharias
Universidade da Madeira, Portugal

RICARDO RODRÍGUEZ
Departamento de Computación, Universidad de Buenos Aires, Argentina

Abstract

Alchourrón devoted his last years to the analysis of the notion of defeasible conditionalization. He developed a formal system capturing the essentials of this notion. His definition of the defeasible conditional is given in terms of strict implication operator and a modal operator *f* which is interpreted as a revision function at the language level. In this paper, we will point out that this underlying revision function is more general than the well known AGM revision [4]. In addition, we will give a complete characterization of that more general kind of revision and will show how permits to unify models of revision given by other authors.

KEY WORDS: defeasible conditional - strict implication - revision function.

Resumen

Alchourrón dedicó sus últimos años al análisis de la noción de condicionalización derrotable. Desarrolló un sistema formal que captura los aspectos esenciales de esta noción. Su definición de condicional revisable está dada en términos del operador de implicación estricta y un operador modal *f* el cual es interpretado en el nivel del lenguaje como una función de revisión. En este trabajo señalamos que la función de revisión subyacente es más general que la bien conocida función de revisión de AGM. Además, presentaremos una caracterización completa de una clase de revisión más general y mostraremos cómo nuestro modelo permite unificar los modelos de revisión propuestos por otros autores.

PALABRAS CLAVE: condicional derrotable - implicación estricta - función de revisión.

1. Introduction

In the last years of his life, Carlos Alchourrón published a series of articles on the logic of defeasible conditionals [1,2,3]. In these papers Alchourrón not only provided a philosophical elucidation of the notion of defeasibility but also showed how to apply it in the explication of deontic concepts such as the notion of prima facie duty. The Alchourrón's main contribution to the theory of defeasible conditional has been to marry the concepts of revision and defeasible conditional in his modal system DFT using a revision function at the language level. Thus, he arrived at the new semantic for defeaseble conditional in terms of revision.

His main theses on this subject can be summarized as follows: a) the conditional constructions of ordinary language are often used in such a way that the antecedent α together with a set of assumptions accepted in the context of utterance of the conditional, but not by itself, is a sufficient condition for the consequent β; b) this kind of conditional is defeasible and can be represented by $f\alpha \Rightarrow \beta$, where $\Rightarrow$ is the strict conditional and $f\alpha$ is used to symbolise the joint assertion of α and the set of assumptions that comes with it; c) the operator f is a special *AGM* revision function (see [4]). In addition, he assumes a choice function *Ch* mirrors at the semantic level the operator f, i.e. for each sentence α it selects the worlds in which α and its associated assumptions are true.

In his "Philosophical Foundations of Deontic Logic and the Logic of Defeasible Conditionals" [1], Alchourrón dedicated a section to the relation between defeasible conditionals and the *AGM* functions of theory change. In the section memorably titled "Fusion and Possible Confusion of Logic and Revision", Alchourrón proposed many perspectives to understand defeasible conditionals but here we only consider the characterisation of the Choice connective f as a revision operator. Just as it has been pointed out in [8], this interpretation is not entirely immediate. In the formal presentation, f is a function of a single propositional argument, say α, and returns another proposition, the conjunction of α and its presuppositions. However, according to *AGM* paradigm, revision functions possess two arguments: the theory K to be revised and the input sentence α, and return a theory $K{*}\alpha$. As far as Alchourrón's formalisation goes, he gives no counterpart for the implicit theory K.

In the current work, based on the association of $f\alpha$ with $K{*}\alpha$, we give an identification of the axioms of f and the postulates of revision which make no explicit reference to any theory K. In addition, we give a characterisation of the family of revision functions for an implicit underlying theory K, that do not necessarily verify postulates inclusion and vacuity. These postulates describe the case of revision by consistent information and play a crucial role in the case of conditional theories. We shall dedicate a special passage to this controversial topic below.

Our interpretation is in intimate relation to the work of Gärdenfors and Makinson [11,12], who showed the connection between a nonmonotonic consequence relation and the logic of theory change. Both the non-monotonic consequence relation and Alchourrón's conditional logic were explained in terms of a special revision function that satisfies all the *AGM* revision postulates but not inclusion and vacuity. They also share the necessary assumption of an implicit theory K, which is of an existential nature. It can be proved that the properties of the non monotonic consequence relation are in correspondence with those of Alchourrón's conditional connective (with the proviso that the one is extralinguistic while the other is not).

So far we have commented on Alchourrón's foundational ideas. A first analysis of his ideas has been made in [8] by us. In the following passages we will continue reviewing some aspect of the theory of defeasible conditionals and will dare to give a novel interpretation of the revision function behind Alchourrón's conditionals.

His logic can be considered as an special conditional logic that provide a calculus for a particular class of Belief Functions. In this paper we are specially interested in the formal counterparts of this intuitive notions in terms of the logic of theory change.

The class of revision functions that we consider here are more general than the ones proposed in [4]. One of their representations will be based on sphere systems [14]. A sphere system is a total order among worlds (not necessarily well founded) where the worlds that belong to the same layer are indistinguishable from each other. Besides, each sphere represents a possible theory as the set of formulas that are satisfied by every world that belongs to this sphere. In our approach, we consider a sphere system and one distinguished sphere as representing our preferences and our beliefs. In this context, we have two ways of changing our beliefs: internal or external. If in order to our beliefs we cannot affirm neither the truth of a sentence β nor its negation $\neg\beta$, and we have the necessity for assuming the one or the other, then we have to use our internal preferences. In order to do that, we borrow from Pagnucco [17] the concept of abductive expansion and the nonmonotonic consequence associated to it. On the contrary, if we believe β but we have to do a counterfactual reasoning assuming that $\neg\beta$ is possible and we want to conclude if another sentence γ or its negation $\neg\gamma$ is true, then we use the usual notion of nonmonotonic reasoning such as it appears in [12].

In this paper we show how our more general model of revision permits to unify internal and external revision in a unique one.

Throughout our presentation, we will assume some familiarity with classical logic and with *AGM* theory. In Section 2 we will introduce notational conventions and review the definitions and main results of both *DFT* logic and theory change that will be needed. In Section 3 we will formally present the main results in this paper. In Section 4 we will give some final comment and will mention some lines of further research. Finally, in the appendix we will present the proofs of lemmas and theorems.

2. Background

2.1. The Logic DFT [1]

Alchourrón's modal conditional logic is based on a propositional language L augmented with an S5-necessity operator $\Box$ and a revision operator f, which is in fact another modality. We will refer to this modal language with L_{DFT}. Alchourrón based his construction on the idea that in a *defeasible conditional* the antecedent is a *contributory* condition (i.e. a necessary condition of a sufficient condition) of its consequent. Hence, he defined a defeasible conditional $\alpha > \beta$ meaning that the antecedent α jointly with the set of assumptions that comes with it is a sufficient condition for the consequent β. In order to represent in the object language the joint assertion of the proposition expressed by a sentence α and the set of assumptions that comes with it, he used a *revision operator f*. For example, if $\alpha_1 \ldots \alpha_n$ are the assumptions associated with α, then $f\alpha$ stands for the joint assertion (conjunction) of α with all the α_i (for all $1 \leq i \leq n$), where α is always one of the conjuncts of $f\alpha$. Although Alchourrón did not explicitly refer to the cardinality of the set of assumptions for a given proposition, this set may well be infinite and $f\alpha$ stands for a nominal of the infinite conjunction.

For reasons that we will see later, he does not allow for the occurrence of an operator f within the scope of another operator f, i.e. his language is flat with respect to this kind of operators. Since L_{DFT} is the standard modal language of S5 augmented with f, the S5-possibility operator $\Diamond$ and the strict conditional $\Rightarrow$ are defined in terms of $\Box$ as usual:

$$\Diamond\alpha \equiv_{df} \neg\Box\neg\alpha \text{ and } \alpha \Rightarrow \beta \equiv_{df} \Box(\alpha \to \beta)$$

Definition 1. [2] The conditional logic DFT is the smallest set $S \subseteq L_{DFT}$ such that S contains classical propositional logic and the following axiom schemata, and is closed under the following rules of inference:

K $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$.
T $\Box\alpha \to \alpha$.

[1] We have partially borrowed this section from [7].

_4_ $\Box\alpha \rightarrow \Box\Box\,\alpha$.
_5_ $\alpha \rightarrow \Box\Diamond\alpha$.
_f.1_ $(f\,\alpha \rightarrow \alpha)$. *(Expansion)*
_f.2_ $(\alpha \equiv \beta) \rightarrow (f\,\alpha \equiv fB)$. *(Extensionality)*
_f.3_ $\Diamond\,\alpha \rightarrow \Diamond\,f\,\alpha$. *(Limit Expansion)*
_f.4_ $(f(\alpha \vee \beta) \leftrightarrow f\alpha) \vee (f(\alpha \vee \beta) \leftrightarrow f\,\beta) \vee (f(\alpha \vee \beta) \leftrightarrow (f\alpha \vee f\beta))$ *(Hierarchical Ordering)*
Nes *From* α *infer* $\Box\,\alpha$.
_MP_ *From* $\alpha \rightarrow \beta$ *and* α *infer* β.

Axioms ***K*****, T, 4** and **5** define to S5, and **f.1-f.4** are constraints imposed on the revision operator *f*. Condition **f.1** is in fact the characteristic axiom ***T*** of standard modal systems. As an axiom constraining *f* it is quite natural since it states that $f\alpha$, stands for the conjunction of α and its presuppositions implies one of the conjuncts: α. **f.2** asserts that equivalent sentences have equivalent presuppositions. **f.3** links the two modalities. It ensures the existence of consistent presuppositions for any sentence that is not a contradiction. **f.4** asserts that the presuppositions of a disjunction are either the presuppositions of one of the disjuncts, or else the disjunction of the presuppositions of each of the disjuncts. In a forward reading it implies that *f* is a normal modality, in the sense that it satisfies the characteristic axiom ***K*** (notice that $\vdash_{DFT} f(\neg\,\alpha) \rightarrow \neg(f\alpha))$.

Alchourrón gave a formal semantic interpretation of the language L_{DFT} based on standard non-relational S5-models.

Definition 2. *A model for* L_{DFT} *is* $M_{DFT} = \langle W, Ch, [\ ] \rangle$ *where W is a non empty set, the valuation function [ ] maps Var (the set of propositional variables) into P(W), and* $Ch\colon L \mapsto P(W)$ *is a selection function such that for each sentence* α, β *of* L_{DFT}

_Ch.1_ $Ch(\alpha) \subseteq [\alpha]$.
_Ch.2_ *If* $[\alpha] = [\beta]$ *then* $Ch(\alpha) = Ch(\beta)$.
_Ch.3_ *If* $[\alpha] \neq \varnothing$ *then* $Ch(\alpha) \neq \varnothing$.
_Ch.4_ $Ch(\alpha \vee \beta) \in \{Ch(\alpha), Ch(\beta), Ch(\alpha) \cup Ch(\beta)\}$.

The selection function *Ch* is proposed as the semantic counterpart of the syntactic revision operator. The choice of the selection function for each sentence α: $Ch(\alpha)$, are the worlds in which α and its assumptions are true, i.e. the worlds in which $f\alpha$ is true.

$$[\,f\alpha\,] = Ch(\alpha).$$

The four constraints on Ch are in exact correspondence with the four on f. In particular, **Ch.3** reflects that every consistent proposition must contain some chosen elements.

A DFT frame $\langle W,Ch\rangle$ is the set of all DFT models having W and Ch. Satisfaction of a modal formula at world w in a model $M=\langle W,Ch,[\ ]\rangle$ is given by:

$M \models_w \alpha$	iff	w$\in$ $[\alpha]$ for atomic sentence α.
$M \models_w \neg\alpha$	iff	not $M \models_w \alpha$.
$M \models_w \alpha \wedge \beta$	iff	$M \models_w \alpha$ and $M \models_w \beta$.
$M \models_w \Box\alpha$	iff	$[\alpha]=W$.
$M \models_w f\alpha$	iff	w $\in$ $Ch(\alpha)$.

The derived satisfaction conditions for the connectives $\Diamond$ and $\Rightarrow$ are:

$M \models_w \alpha \Rightarrow \beta$ iff $[\alpha] \subseteq [\beta]$.
$M \models_w \Diamond\alpha$ iff there is some $v \in W$ such that $v \in [\alpha]$.

Truth in a model $M=\langle W,Ch,[\ ]\rangle$ is truth at every point:

$M \models \alpha$ iff $M \models_w \alpha$ for every $w \in W$

Truth in a frame $\langle W,Ch\rangle$ is truth at every model $\langle W,Ch,[\ ]\rangle$.

$\langle W,Ch\rangle \models \alpha$ iff $\langle W,Ch,[\ ]\rangle \models \alpha$ for all valuation functions [ ].

A set of formulas Γ forces a formula α in the context of a model $M=\langle W,Ch,[\ ]\rangle$, noted as $\Gamma \models_M \alpha$ if and only if every world $w \in W$ if $M \models_w \Gamma$ then $M \models_w \alpha$.

Alchourrón proves that his semantic and axiomatic presentations coincide.

Observation 3. [2] *For any,* $\alpha \in L_{DFT}$, $\models_{DFT} \alpha$ iff $\vdash_{DFT} \alpha$.

2.2. AGM

The idea of belief revision is to describe how the belief state of a rational agent should change when accepting new information. The problem arises when the new piece of information is inconsistent with the agent's beliefs. Some of the beliefs have to be abandoned to maintain consistency, and a rational choice has to be made in order to select which ones.

The AGM paradigm characterizes the revision process in some different ways. In the seminal paper [4], a constructive characterization called partial meet revision was proposed. It consists in choosing the maximal subsets of the belief set which do not imply the new belief and selecting some of them. The intersection of the selected sets is taken, forming a subset that is consistent with the new formula. Partial meet revision is defined as follows:

Formal preliminaries: In the AGM account the beliefs of a rational agent are represented by a belief set K, which is a set of sentences in a language L closed under logical consequence Cn, where Cn satisfies: A $\subseteq Cn$(A), $Cn(Cn$(A)) $\subseteq Cn$(A) and Cn(A) $\subseteq Cn$(B) if A $\subseteq$ B, as well as supraclassicality, deduction and compactness.

We use $\vdash \alpha$ as an alternative notation for $\alpha \in Cn(\emptyset)$, {A} $\vdash \alpha$ for $\alpha \in Cn(\{A\})$, $\alpha \vdash \beta$ for $\beta \in Cn(\{\alpha\})$. $\{K_\perp\}$ denotes the inconsistent belief set. $K + \alpha$ denotes the expansion of K by α and is defined by $K + \alpha = Cn\ (K \cup \{\alpha\})$.

One of the major achievements of AGM theory is the characterization of revision functions in terms of a set of intuitively reasonable postulates [4]. The six basic AGM postulates for revision are:

Closure	$K * \alpha$ is a belief set.
Success	$\alpha \in K * \alpha$.
Inclusion	$K * \alpha \subseteq K + \alpha$.
Vacuity	If $K \nvdash \neg\alpha$, then $K + \alpha \subseteq K * \alpha$.
Consistency	If $\nvdash \neg\alpha$ then $K * \alpha \neq K_\perp$.
Extensionality	If $\vdash \alpha \leftrightarrow \beta$, then $K * \alpha = K * \beta$.

The supplementary AGM postulates are as follows:

Superexpansion	$K * (\alpha \wedge \beta) \subseteq (K * \alpha) + \beta$.
Subexpansion	If $K * \alpha \nvdash \neg\beta$, then $(K * \alpha) + \beta \subseteq K * (\alpha \wedge \beta)\}$.

An operator * that satisfies closure, success, inclusion, vacuity, consistency and extensionality, is called a partial meet revision. In addition, if * also satisfies superexpansion and subexpansion then it is called transitively relational partial meet revision.

The following postulates will be useful in the following sections:

Disjunctive overlap	$(K * \alpha) \cap (K * \beta) \subseteq K * (\alpha \vee \beta)$.
Disjunctive inclusion	If $K * (\alpha \vee \beta) \nvdash \neg\alpha$, then $K * (\alpha \vee \beta) \subseteq K * \alpha$.
Disjunctive factoring	Either $K * (\alpha \vee \beta) = K * \alpha$, or $K * (\alpha \vee \beta) = K * \alpha$, or $K * (\alpha \vee \beta) = (K * \alpha) \cap (K * \beta)$.

Observation 4. [9] *Let K be a belief set and $*$ be an operator for K that satisfies closure, success, inclusion, vacuity, consistency, and extensionality. Then:*

1. *$*$ satisfies disjunctive overlap if and only if it satisfies superexpansion.*
2. *$*$ satisfies disjunctive inclusion if and only if it satisfies subexpansion.*
3. *$*$ satisfies both disjunctive overlap and disjunctive inclusion, if and only if it satisfies disjunctive factoring.*

2.2.1. Epistemic Entrenchment

The notion of epistemic entrenchment for theories was introduced by Gärdenfors [9] to define the properties that an order between sentences of the language should satisfy. Gärdenfors proposed the following set of axioms:

If $\alpha \leq \beta$ and $\beta \leq \delta$, then $\alpha \leq \delta$. (transitivity)
If $\alpha \vdash \beta$, then $\alpha \leq \beta$. (dominance)
$\alpha \leq (\alpha \wedge \beta)$ or $\beta \leq (\alpha \wedge \beta)$. (conjunctiveness)
If $K \nvdash \bot$, then $\alpha \notin K$ if and only if $\alpha \leq \beta$ for all β. (minimality)
If $\beta \leq \alpha$ for all $\beta\}$, then $\vdash \alpha\}$. (maximality)

The connections between orders of epistemic entrenchment and Revision is stabilized by means of the following equivalences [10,16,20]:

$(C \leq)$ $\alpha \leq \beta$ if and only if: If $\alpha \in K * \neg(\alpha \wedge \beta)$ then $\beta \in K * \neg(\alpha \wedge \beta)$.

(EBR) $\beta \in K * \alpha$ if and only if either $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$ or $\alpha \vdash \bot$.

Theorem 5. *Let $\leq$ be an entrenchment ordering on a consistent belief set K that satisfies transitivity, dominance, conjunctiveness, minimality and maximality. Furthermore, let $*$ be an operation on K defined via condition (EBR) from $\leq$. Then $*$ satisfies closure, success, inclusion, vacuity, consistency, extensionality, disjunctive factoring, and $(C \leq)$ also holds.*

Theorem 6. *Let $*$ be an operation on a consistent belief set K that satisfies closure, success, inclusion, vacuity, consistency, extensionality, and disjunctive factoring. Furthermore, let $\leq$ be the relation defined from $*$ by condition $(C \leq)$. Then $\leq$ satisfies transitivity, dominance, conjunctiveness and maximality, and (ERB) also holds.*

2.2.2. Semantic

The semantic for AGM is based on possible worlds models. A proposition (set of possible worlds) can represent either a belief set or an input sentence. The belief set K can be replaced, as a belief state representation, by $[K]$ we indicate the set of worlds that contain K. Similarly, each sentence can be represented by the set $[\alpha] = [Cn(\{\alpha\})]$.

The *Grove's sphere-system* [14] makes use of a system of concentric spheres around $[K]$. Intuitively, each sphere represents a degree of closeness or similarity to $[K]$. Revising by α is to take the closest α-worlds with respect to $[K]$.

Definition 7. [14] *$\$ \subseteq P(\Omega)$ is a system of spheres centred on $[K] \subseteq \Omega$ (the set of all possible worlds) for if and only if it satisfies:*

- **(S1)** $\$$ is totally ordering by $\subseteq$; that is, if $G, G' \in \$$, then $G \subseteq G'$ or $G' \subseteq G$.
- **(S2)** $[K]$ is the $\subseteq$–minimun of $\$$.
- **(S3)** Ω is the $\subseteq$–maximun of $\$$.
- **(S4)** If $[\alpha] \cap \cup\$ \neq \emptyset$, then $S_\alpha = \cap\{G \in \$: G \cap [\alpha] \neq \emptyset\} \in \$$.

Theorem 10. [14] *Let K be a belief set and * an operator for K. Then the following conditions are equivalent:*

*a) * satisfies closure, inclusion, vacuity, success, consistency, extensionality, subexpansion, and superexpansion.*
b) There exists a system of spheres centred on $[K]$ such that for all consistent α, $[K * \alpha] = S_{\alpha'} \cap [\alpha]$

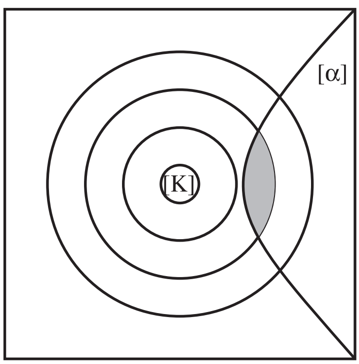

Figure 1. Revision of *K* by α

2.3. Abductive Expansion

In [18] and [17] a new belief change function, *abductive expansion*, is presented. Unlike the AGM expansion (consisting in $K + \alpha = Cn\ (K \cup \{\alpha\})$), the agent incorporates a justification or explanation of the new belief together with the new information. The justification is the "abduction" of a formula and it is defined as follows:

Definition 9. *An abduction of a formula α with respect to a domain theory Γ is a formula β such that:*

1. $\Gamma \cup \{\beta\}) \vdash \alpha$
2. $\Gamma \cup \{\beta\}) \nvdash \bot$

Definition 10. *$K \oplus \alpha$ is an abductive expansion of K with respect to α iff:*

$$K \oplus \alpha = \begin{cases} K + \beta & \textit{for some abduction } \beta \textit{ of a formula } \alpha \textit{ wrt } \Gamma. \\ K & \textit{if no such } \beta \textit{ exists.} \end{cases}$$

The postulates that characterize abductive expansion are:

Closure $K \oplus \alpha$ is a belief set.

Limited Success If $\neg\alpha \notin K$, then $\alpha \in K \oplus \alpha$.

Inclusion $K \subseteq K \oplus \alpha$.

Vacuity If $\neg\alpha \in K$, then $K \oplus \alpha = K$.

Consistency If $\neg\alpha \notin K$ then $\neg\alpha \notin K \oplus \alpha$.

Extensionality If $\vdash \alpha \leftrightarrow \beta\}$, then $K \oplus \alpha = K \oplus \beta$.

For the supplementary level, disjunctive factoring is added:

Disjunctive factoring

Either $K \oplus (\alpha \vee \beta) = K \oplus \alpha$, or $K \oplus (\alpha \vee \beta) = K \oplus \beta$, or
$K \oplus (\alpha \vee \beta) =(K \oplus \alpha) \cap (K \oplus \beta)$.

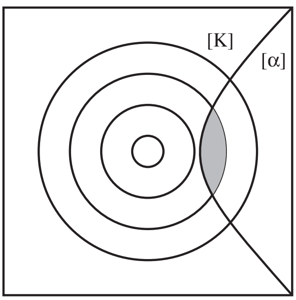

Figure 2. Abductive expansion of *K* by α

In order to present the semantic, we will also use the Grove's systems of spheres. The construction is very similar to AGM belief revision. The difference arises in AGM revision where $[K]$ is the innermost sphere; in abducting expansion this condition is not anymore valid, i.e. a system of spheres can exist inside $[K]$.

2.4. From DFT to AGM

In order to conclude this background, a first and well known connection between AGM and DFT will be given here. In this sense, we have to note that the DFT models are able to define a belief operator in the following way:

Observation 11. *Given a belief set K, we can take the conditional theory:*

$$\mathrm{Th}_K = \{\top > \alpha \mid \alpha \in K\} \cup \{\neg(\top > \alpha) \mid \alpha \notin K\}.$$

Then to check if β belong to the revised belief set $K^*\alpha$ will correspond, in a DFT model, to evaluate the conditional $\alpha > \beta$ at all worlds satisfying the theory Th_K, that is, at all worlds whose corresponding belief are K. More formally:

Theorem 12. Let K and $*$ be a belief set and a partial meet revision on K, respectively. Then there exists a DFT model M^* such that for each $\alpha, \beta \in L$,

$$\beta \in K * \alpha \text{ if and only if } \mathrm{Th}_K \models_{M^*} \alpha > \beta$$

In this work, we are interested in a more general connection between AGM and DFT such as it is shown in the next section.

3. DFT and its Theory Change Associated

3.1. DFT as Belief Revision

In this section we will define three different models to capture the idea of defeasible conditional DFT.

3.1.1. Axiomatic

In order to characterize the operator f as a belief revision operator, we can "translate" (in the sense of intuitions) the axioms for the operator (see Definition 1): $f.1$ claims that α is part of $K * \alpha$, in other words, the success postulate. In the same way *f*.2 corresponds to extensionality, *f*.3 to consistency and *f*.4 to disjunctive factoring. The postulate closure expresses that the result of a revision is always a theory, hence it is implicit in the DFT definition.

With this translation we can define a first model of belief revision to capture the DFT ideas:

Definition 13. *Model0: $\otimes$ satisfies closure, success, consistency, extensionality and disjunctive factoring.*

We can extend the Model0 in the following way: Alchourrón claims that the Choice function must be understood as "α and the presuppositions for α". But what happens in the case where α is consistent with our corpus of beliefs? In that case, there is no reason to eliminate beliefs in the revision process. We capture this intuition in this extension of Model0:

Definition 14. *Model1: $\otimes$ satisfies closure, success, preservation, consistency, extensionality, disjunctive factoring and*

Preservation If $\neg\alpha \notin K$, then $K \subseteq K \otimes \alpha$.

Finally, if the sentence α is already in our corpus of belief, we can assume that all the presuppositions for α are also included in it; consequently, there is no reason to perform any change:

Definition 15. *Model2: $\otimes$ satisfies closure, success, consistency, extensionality, disjunctive factoring and*

Vacuity 2 If $\alpha \in K$, then $K \otimes \alpha = K$.[2]

The arisen question is the relation between the model proposed and AGM belief revision. Model0 uses all the AGM postulates, except inclusion and vacuity. These postulates are the only postulates in the AGM axiomatic that refer to the corpus of belief K, and consequently, the postulates related to the rationality criteria of "minimal change" (for a deep analysis about the relation between these postulates and minimal change see [21]).

In our Model0, no corpus of belief is specified. In Model1 appears the first constraint related to the actual beliefs by means of the preservation postulate. It is easy to prove that the AGM vacuity postulate follows

[2] Preservation is redundant in this axiomatic.

from preservation and success. Model1 is related to minimal change since it constrains the function to conserve all the previous beliefs when they are consistent with the new belief. On the other hand Model2 establishes an upper limit for the revision when the new belief is already believed.

The AGM inclusion postulate cannot be derived in our models. The reason is simple. Inclusion establishes a very strong condition in the revision process: Revise by α cannot add more beliefs than the included in the consequences of $K \cup \{\alpha\}$. This constraint does not permit to include "the presuppositions of α". Consequently, we exclude it from our models.

3.1.2. Constructing DFT-related revision functions

We will construct DFT-related revision functions using epistemic entrenchment. In standard AGM epistemic entrenchment, *minimality* claims that all the sentences that are not in K are all in the bottom of the ordering. In order to construct DFT-related revision function, the *minimality* postulate will be discarded. We can construct a DFT-related revision function Model0 by means of ($C \leq$) and (EBR):

Theorem 16. *Let ≤ be an entrenchment ordering on a consistent belief set K that satisfies transitivity, dominance, conjunctiveness and maximality. Furthermore, let ⊗ be an operation on K defined via condition* (EBR) *from ≤. Then ⊗ satisfies closure, success, consistency, extensionality, disjunctive factoring, and* ($C \leq$) *also holds.*

Theorem 17. *Let ⊗ be an operation on a consistent belief set that satisfies closure, success, consistency, extensionality and disjunctive factoring. Furthermore, let ≤ be the relation defined from * by condition* ($C \leq$)*. Then ≤ satisfies transitivity, dominance, conjunctiveness and maximality, and* (EBR) *also holds.*

In order to construct DFT-related revision function Model1 and Model2 we need a new postulate for the entrenchment ordering:

If $\alpha \notin K$ and $\beta \in K$, then $\alpha < \beta$. (primacy)

Primacy, as minimality, establishes that the sentences that are not in K are less entrenched than the sentences that are in K. However, primacy does not force that all the sentences outside K are equally entrenched.

Theorem 18. *Let ≤ be an entrenchment ordering on a consistent belief set K that satisfies transitivity, dominance, conjunctiveness and maximality. Furthermore, let ⊗ be an operation on K defined via condition*

(EBR) *from* $\leq$. *Then* $\otimes$ *satisfies closure, success, preservation, consistency, extensionality, disjunctive factoring, and* $(C \leq)$ *also holds.*

Theorem 19. *Let* $\otimes$ *be an operation on a consistent belief set* K *that satisfies closure, success, preservation, consistency, extensionality and disjunctive factoring. Furthermore, let* $\leq$ *be the relation defined from* $\otimes$ *by condition* $(C \leq)$. *Then* $\leq$ *satisfies transitivity, dominance, conjunctiveness, maximality and primacy, and* (EBR) *also holds.*

In DFT-related revision Model2, the postulate vacuity 2 is incorporated. (EBR) is not enough to guarantee that K will be unchanged if $\alpha \in K$. Consequently we must modify the identity as follows:

$$(EBR\_2)\ \beta \in K \otimes \alpha \textit{ iff } \begin{cases} (\text{either } \alpha \to \neg\beta < \alpha \to \beta \text{ or } \alpha \vdash \bot) & \text{and } \alpha \notin K. \\ \beta \in K & \text{and } \alpha \in K. \end{cases}$$

Theorem 20. *Let* $\leq$ *be an entrenchment ordering on a consistent belief set* K *that satisfies transitivity, dominance, conjunctiveness and maximality. Furthermore, let* $\otimes$ *be an operation on* K *defined via condition* (EBR_2) *from* $\leq$. *Then* $\otimes$ *satisfies closure, success, preservation, vacuity 2, consistency, extensionality, disjunctive factoring, and* $(C \leq)$ *also holds.*

Theorem 21. *Let* $\otimes$ *be an operation on a consistent belief set* K *that satisfies closure, success, vacuity 2, consistency, extensionality and disjunctive factoring. Furthermore, let* $\leq$ *be the relation defined from* $\otimes$ *by condition* $(C \leq)$. *Then* $\leq$ *satisfies transitivity, dominance, conjunctiveness, maximality and primacy, and* (EBR_2) *also holds.*

3.2. DFT as Abductive Expansion

Another way to interpret the DFT logic as a change function is as Abductive Expansion. The key idea is that incorporating "α and the presuppositions for α" is more than simply expanding by α (i.e., $K+\alpha = Cn(K \cup \{\alpha\})$). These presuppositions constitute the *abductive part* of the change and will be incorporated. This interpretation matches the axioms for the operator f, but with an additional constraint: α must be consistent with our corpus of belief, otherwise, due to the postulate of vacuity, f.1 (success) does not hold. Hence, to interpret DFT as abductive expansion we have two ways: 1. considering this model just for sentences that are consistent with our corpus of belief, or, 2. contract first by the negation of the new sentence and then make the abductive expansion.

Definition 22. *Model4: $\otimes$ satisfies the abductive expansion postulates closure, limited success, inclusion, vacuity, consistency, extensionality and disjunctive factoring.*

Several ways to construct Model4 functions can be found in [18] and [17]. In particular, for epistemic entrenchment Pagnucco [17, pp 122-125] defines:

Definition 23. $\leq$ is an abductive entrenchment ordering if it satisfies transitivity, dominance, conjunctiveness and

When $K \nvdash \perp$, $\alpha \in K$ if and only if $\beta \leq \alpha$ for all β}. (maximality 2)

The connection to the axiomatic is given by the following equivalences:

($C2 \leq$) $\alpha \leq \beta$ if and only if either $\alpha \notin K \otimes \neg(\alpha \wedge \beta)$ or $K \vdash \alpha \wedge \beta$.

(EBAE) $\beta \in K \otimes \alpha$ if and only if either $\beta \in K$ or both $\neg\alpha \notin K$ or $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$.

Theorem 24. *Let $\leq$ be an abductive entrenchment ordering on a consistent belief set K. Furthermore, let $\otimes$ be an operation on K defined via condition (EBAE) from $\leq$. Then $\otimes$ satisfies the abductive expansion postulates closure, limited success, inclusion, vacuity, consistency, extensionality and disjunctive factoring, and ($C2 \leq$) also holds.*

Theorem 25. *Let $\otimes$ be an operation on a consistent belief set K that satisfies the abductive expansion postulates closure, limited success, inclusion, vacuity, consistency, extensionality and disjunctive factoring. Furthermore, let $\leq$ be the relation defined from $\otimes$ by condition ($C2 \leq$). Then $\leq$ is an abductive entrenchment ordering, and (EBAE) also holds.*

4. Conclusions and future works

We have characterized the revision and expansion operations behind the operator *f* in DFT logic in terms of postulates, epistemic entrenchment and sphere systems. Three new kinds of revision function have been presented. Besides, we have used to explain the Pagnucco's abductive model in terms of an operation of change when the input does not contradict what is already in the original theory, which is called *addition* by Rott in [19].

We believe our interpretation is particularly appealing because it relates defeasible conditions with a simpler operator satisfying all of AGM postulates except inclusion and vacuity, which, according to Rott's elucidations, lack of these postulates is a sufficient condition to avoid Gärdenfors impossibility results.[3] In this work we have not addressed ourselves to avoid this problem of change functions of conditional theories. However, we think this approach is appropriate and it deserves more research. We leave its investigation for a future paper.

Appendix. Proofs

Previous Lemmas [9,15]

Lemma 26.
Let $\leq$ be an entrenchment ordering. Then:
(a) *If $\nvdash \alpha$ and $\vdash \beta$, then $\alpha < \beta$.*
(b) *If $\alpha < \beta$ then $\alpha \leq \alpha \wedge \beta$.*
(c) *$\alpha \to \beta < \alpha \to \neg\beta$ if and only if $\neg\alpha < \alpha \to \neg\beta$.*
(d) *If $\vdash \alpha \leftrightarrow \alpha'$ and $\vdash \beta \leftrightarrow \beta'$, then: $\alpha \leq \beta$ if and only if $\alpha' \leq \beta'$. (intersubstitutivity)*

Proof of Theorem 16
Closure: Let $\varepsilon \in L$. Then, by compactness of the underlying logic, there is a finite subset $\{\beta_1,...,\beta_n\} \subseteq L$, such that $\{\beta_1,...,\beta_n\} \vdash \varepsilon$. We must prove that if $\{\beta_1,...,\beta_n\} \subseteq K \otimes \alpha$, then $\beta_1 \wedge ... \wedge \beta_n \in K \otimes \alpha$ and $\varepsilon \in K \otimes \alpha$. If $\alpha \vdash \bot$, then it follows trivially from (EBR) that $\beta_1 \wedge ... \wedge \beta_n \in K \otimes \alpha$ and $\varepsilon \in K \otimes \alpha$. Let $\alpha \nvdash \bot$. Then:

1. **[Part 1.]** We are going to show that $\beta_1 \wedge ... \wedge \beta_n \in K \otimes \alpha$. For this purpose we are going to prove that if $\beta_1 \in K \otimes \alpha$ and $\beta_2 \in K \otimes \alpha$ then $\beta_1 \wedge \beta_2 \in K \otimes \alpha$. The rest follows by iteration of the same procedure. It follows from $\beta_1 \in K \otimes \alpha$ by (EBR) that $(\alpha \to \neg\beta_1) < (\alpha \to \beta_1)$. Then by **Lemma 26** b and c, $\neg\alpha < (\alpha \to \beta_1)$. Then it follows from $\beta_2 \in K \otimes \alpha$ that $\neg\alpha < (\alpha \to \beta_2)$. By conjunctiveness, either $(\alpha \to \beta_1) \leq ((\alpha \to \beta_1) \wedge (\alpha \to \beta_2))$ or $(\alpha \to \beta_1) \leq ((\alpha \to \beta_1) \wedge (\alpha \to \beta_2))$. Equivalently by intersubstitutivity either $(\alpha \to \beta_1) \leq (\alpha \to (\beta_1 \wedge \beta_2))$ or $(\alpha \to \beta_2) \leq (\alpha \to (\beta_1 \wedge \beta_2))$. In the first case, we use transitivity and $\neg\alpha < (\alpha \to \beta_1)$ to obtain $\neg\alpha < (\alpha \to (\beta_1 \wedge \beta_2))$ and in the second one we use $\neg\alpha < (\alpha \to \beta_2)$ to obtain the same result. It follows that $\beta_1 \wedge \beta_2 \in K \otimes \alpha$.

[3] He has shown in [13] that AGM revision operator on a conditional language are incompatible with the Ramsey test for interpreting conditionals.

2. [Part 2.] By repeated use of **Part 1**, we know that $\{\beta_1 \wedge \ldots \wedge \beta_n \in K \otimes \alpha$. Let $\vdash \beta \leftrightarrow \beta_1 \wedge \ldots \wedge \beta_n$. We also have $\vdash \beta \rightarrow \varepsilon$, then by (EBR) $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$. Since $\vdash (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \varepsilon)$ and $\vdash (\alpha \rightarrow \neg\varepsilon) \rightarrow (\alpha \rightarrow \neg\beta)$, dominance yields $(\alpha \rightarrow \beta) \leq (\alpha \rightarrow \varepsilon)$ and $(\alpha \rightarrow \neg\varepsilon) \leq (\alpha \rightarrow \neg\beta)$. We can apply transitivity to $(\alpha \rightarrow \neg\varepsilon) \leq (\alpha \rightarrow \neg\beta)$, $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$ and $(\alpha \rightarrow \beta) \leq (\alpha \rightarrow \varepsilon)$ to obtain $(\alpha \rightarrow \neg\varepsilon) < (\alpha \rightarrow \varepsilon)$. Hence by (*EBR*), $\varepsilon \in K \otimes \alpha$.

Success: If $\alpha \vdash \bot$, then it follows trivially from (EBR) that $\alpha \in K \otimes \alpha$. Let $\alpha \nvdash \bot$. Then by **Lemma 26** $\neg\alpha < (\neg\alpha \vee \alpha)$ or equivalently by intersubstitutivity $(\alpha \rightarrow \neg\alpha) < (\alpha \rightarrow \alpha)$. Hence by (*EBR*), $\alpha \in K \otimes \alpha$.

Consistency: Let $\alpha \nvdash \bot$. Assume by reduction that $\bot \in K \otimes \alpha$ and $\alpha \nvdash \bot$. Then by (*EBR*), $(\alpha \rightarrow \neg\bot) < (\alpha \rightarrow \bot)$. Then by intersubstitutivity $\top < \neg\alpha$. Contradiction by dominance.

Extensionality: Let $\vdash \alpha \leftrightarrow \beta$. If $\alpha \in K$, then $\beta \in K$ and the rest follows trivially by (*EBR*). If $\alpha \vdash \bot$ then $\beta \vdash \bot$, hence by (*EBR*), $K \otimes \alpha = K \otimes \beta$. By intersubstitutivity it follows for all δ that $(\alpha \rightarrow \neg\delta) \leq (\beta \rightarrow \neg\, \delta)$ and $(\alpha \rightarrow \delta) \leq (\beta \rightarrow \delta)$. Hence by transitivity $(\alpha \rightarrow \neg\delta) < (\alpha \rightarrow \delta)$ if and only if $(\beta \rightarrow \neg\delta) < (\beta \rightarrow \delta)$; hence $K \otimes \alpha = K \otimes \alpha'$.

Disjunctive factoring: If $\alpha \in K$, then $\alpha \vee \beta \in K$, hence $K \otimes (\alpha \vee \beta) = K \otimes \alpha$. If $\beta \in K$, then $\alpha \vee \beta \in K$, hence $K \otimes (\alpha \vee \beta) = K \otimes \beta$. Let $\alpha \notin K$ and $\beta \notin K$. Then $\nvdash \alpha$ and $\nvdash \beta$. We have three subcases:

a) $\neg\alpha < \neg\beta$. Then $\nvdash \neg\alpha$. We will prove that $K \otimes (\alpha \vee \beta) = K \otimes \alpha$. For one direction let $\delta \in K \otimes \alpha$. It follows by (*EBR*) that $(\alpha \rightarrow \neg\delta) < (\alpha \rightarrow \delta)$. Then by **Lemma 26** b and c, $\neg\alpha < (\alpha \rightarrow \delta)$. It follows by $\neg\alpha < \neg\beta$ and **Lemma 26** that $\neg\alpha \leq (\neg\alpha \wedge \neg\beta)$. Since dominance yields $\neg\beta < (\beta \rightarrow \delta)$, we use transitivity to obtain both $(\neg\alpha \wedge \neg\beta) < (\alpha \rightarrow \delta)$ and $(\neg\alpha \wedge \neg\beta) < (\beta \rightarrow \delta)$. Dominance and conjunctiveness yield $(\neg\alpha \wedge \neg\beta) < ((\alpha \vee \beta) \rightarrow \delta)$. Hence $\delta \in K \otimes (\alpha \vee \beta)$. For the other direction, let $\delta \in K \otimes (\alpha \vee \beta)$. It follows by $\neg\alpha \leq (\neg\alpha \wedge \neg\beta)$ that $\nvdash (\neg\alpha \wedge \neg\beta)$; then by (*EBR*), $(\neg\alpha \wedge \neg\beta) < ((\alpha \vee \beta) \rightarrow \delta)$. By dominance $((\alpha \vee \beta) \rightarrow \delta) \leq (\alpha \rightarrow \delta)$. Transitivity yields $\neg\alpha < (\alpha \rightarrow \delta)$, hence $\delta \in K \otimes \alpha$.

b) $\neg\beta < \neg\alpha$: Equivalently to case \bf (a); $K \otimes (\alpha \vee \beta) = K \otimes \beta$.

c) $\neg\alpha \leq \neg\beta$. Then $\neg\alpha \leq \neg\beta \leq (\neg\alpha \wedge \neg\beta)$. Then $\delta \in K \otimes \alpha \cap K \otimes \beta$ iff (by (*EBR*)) $\neg\alpha < (\alpha \rightarrow \delta)$ and $\neg\beta < (\alpha \rightarrow \delta)$ iff (by transitivity) $(\neg\alpha \wedge \neg\beta) < (\alpha \rightarrow \delta)$ and $(\neg\alpha \wedge \neg\beta) < (\alpha \rightarrow \delta)$ iff (by dominance and conjunctiveness) $(\neg\alpha \wedge \neg\beta) < ((\alpha \vee \beta) \rightarrow \delta)$ iff (by (*EBR*)) $\delta \in K \otimes (\alpha \vee \beta)$.

(C $\leq$) For the first direction, let $\alpha \leq \beta$ and let $\alpha \in K \otimes \neg(\alpha \wedge \beta)$. If $\alpha \in K$, then by primacy $\beta \in K$, hence $\beta \in K \otimes \neg(\alpha \wedge \beta)$. Let $\alpha \notin K$. We have two subcases according to (*EBR*): If $\neg(\alpha \wedge \beta) \vdash \bot$, it follows trivially from (*EBR*) that $\beta \in K \otimes \neg(\alpha \wedge \beta)$. Let $\neg(\alpha \wedge \beta) \nvdash \bot$, then $(\neg(\alpha \wedge \beta) \rightarrow \neg\alpha) < (\neg(\alpha \wedge \beta) \rightarrow \alpha)$, then by intersubstitutivity, $(\beta \vee \neg\alpha) < \alpha$. By dominance, $\beta \leq (\beta \vee \neg\alpha)$, then it follows by transitivity that $\beta < \alpha$. Contradiction.

The other direction can be proved by showing that **a)** if $\beta < \alpha$, then $\alpha \in K \otimes \neg(\alpha \wedge \beta)$ and **b)** if $\beta < \alpha$, then $\beta \notin K \otimes \neg(\alpha \wedge \beta)$.

a) We can do this by showing $\neg(\alpha \wedge \beta) \rightarrow \neg\alpha < \neg(\alpha \wedge \beta) \rightarrow \alpha$, or equivalently, $\beta \vee \neg\alpha < \alpha$. Suppose for reduction that this is not the case. Then $\alpha \leq \beta \vee \neg\alpha$. Since $\alpha \leq \alpha$, conjunctiveness yields $\alpha \leq \alpha \wedge (\beta \vee \neg\alpha)$, hence $\alpha \leq \alpha \wedge \beta$, so that by transitivity $\alpha \leq \beta$, contrary to the conditions.

b) Suppose to the contrary that $\beta < \alpha$ and $\beta \in K \otimes \neg(\alpha \wedge \beta)$. There are two cases according to (*EBR*): **(b1)** $\vdash \alpha \wedge \beta$. Then $\vdash \beta$, hence by maximality $\alpha \leq \beta$, contrary to the conditions. **(b2)** $\neg(\alpha \wedge \beta) \rightarrow \neg\beta < \neg(\alpha \wedge \beta) \rightarrow \beta$, or equivalently by intersubstitutivity to $\alpha \wedge \beta < \beta$, from which it follows by transitivity that $\alpha \wedge \beta < \alpha$. We arrive to a contradiction according to conjunctiveness. This concludes the proof. ■

Proof of Theorem 17

Transitivity: Let $\alpha \leq \beta$, $\beta \leq \Gamma$ and $\alpha \in K \otimes \neg(\alpha \wedge \gamma)$. We need to prove $\Gamma \in K \otimes \neg(\alpha \wedge \gamma)$.

a) $\alpha \in K \otimes \neg(\alpha \wedge \beta)\}$. Then by ($C \leq$), $\beta \in K \otimes \neg(\alpha \wedge \beta)$. Then by *closure* $\alpha \wedge \beta \in K \otimes \neg(\alpha \wedge \beta)\}$. It follows by *consistency* and *success* that $\vdash \alpha \wedge \beta$, then $\vdash \beta$. *Closure* yields $\beta \in K \otimes \neg(\beta \wedge \gamma)$. Then by ($C \leq$), $\Gamma \in K \otimes \neg(\beta \wedge \gamma)$. Then by *closure* $\beta \wedge \Gamma \in K \otimes \neg(\beta \wedge \gamma)$. It follows by *consistency* and *success* that $\vdash \beta \wedge \gamma$, then $\vdash \gamma$. Hence by *closure* $\Gamma \in K \otimes \neg(\alpha \wedge \gamma)$.

b) $\alpha \notin K \otimes \neg(\alpha \wedge \beta)$. If $\beta \in K \otimes \neg(\beta \wedge \gamma)\}$, we have proven in part (a) that this implies $\Gamma \in K \otimes \neg(\alpha \wedge \gamma)$. Let $\beta \notin K \otimes \neg(\beta \wedge \gamma)$. We will prove that this is not a valid case: It follows from *closure* that $\nvdash \beta$, then $\neg(\alpha \wedge \beta \wedge \gamma) \nvdash \bot$, from which it follows by *consistency* that $K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$ is consistent. $\neg(\alpha \wedge \beta \wedge \gamma)$ is logically equivalent to $\neg(\alpha \wedge \gamma) \vee (\alpha \wedge \neg\beta)$. It follows by *success* that $\alpha \in K \otimes \alpha \wedge \neg\beta$. Then by *disjunctive factoring* $\alpha \in K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$. In the same way $\neg(\alpha \wedge \beta \wedge \gamma)$ is logically equivalent to $\neg(\alpha \wedge \beta) \vee (\beta \wedge \neg\gamma)$. Then (due to $\alpha \in K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$ and $\alpha \notin K \otimes \neg(\alpha \wedge \beta)$) it follows by

\em disjunctive factoring that $K \otimes \neg(\alpha \wedge \beta \wedge \gamma) = K \otimes (\beta \wedge \neg\gamma)$. Then $\beta \in K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$ and $\neg\Gamma \in K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$. One more time, $\neg(\alpha \wedge \beta \wedge \gamma)$ is logically equivalent to $\neg(\beta \wedge \gamma) \vee (\beta \wedge \neg\alpha)$, and due to $\beta \notin K \otimes \neg(\beta \wedge \gamma)$, it follows by *disjunctive factoring* that $K \otimes \neg(\alpha \wedge \beta \wedge \gamma) = K \otimes (\beta \wedge \neg\alpha)$. Hence $\alpha \in K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$, that contradict the consistency of $K \otimes \neg(\alpha \wedge \beta \wedge \gamma)$.

Dominance: Let $\vdash \alpha \rightarrow \beta$, and $\alpha \in K \otimes \neg(\alpha \wedge \beta)$. Then by *closure* $\beta \in K \otimes \neg(\alpha \wedge \beta)$; hence by $(C \leq)$ $\alpha \leq \beta$.

Conjunctiveness: We have three subcases:

a) $\alpha \notin K \otimes \neg(\alpha \wedge \beta)$. Then by *extensionality* $\alpha \notin K \otimes \neg(\alpha \wedge (\alpha \wedge \beta))$, hence by $(C \leq)$ $\alpha \leq (\alpha \wedge \beta)$.

b) $\beta \notin K \otimes \neg(\alpha \wedge \beta)$. In the same way as in **a)**, $\beta \leq (\alpha \wedge \beta)$.

c) $\alpha \in K \otimes \neg(\alpha \wedge \beta)$ and $\beta \in K \otimes \neg(\alpha \wedge \beta)$. Then by *closure*, $(\alpha \wedge \beta) \in K \otimes \neg(\alpha \wedge \beta)$. Hence by $(C \leq)$, $\alpha \leq (\alpha \wedge \beta)$ and $\beta \leq (\alpha \wedge \beta)$.

Maximality: Let $\beta \leq \alpha$ for all β. Then, in particular $\top \leq \alpha$. Then by (C≤) if $\top \in K \otimes \neg(\alpha \wedge \top)$ then $\alpha \in K \otimes \neg(\alpha \wedge \top)$. Then by *closure* $\alpha \in K \otimes \neg(\alpha \wedge \top)$ that is equivalent by *extensionality* to $\alpha \in K \otimes \neg\alpha$. Hence by *success* and *consistency* $\vdash \alpha$.

(*EBR*): From left to right, let $\beta \in K \otimes \alpha$ and $\nvdash \neg\alpha$, then by *closure* $(\alpha \rightarrow \beta) \in K \otimes \alpha$ and by *consistency* and *success* $(\alpha \rightarrow \neg\beta) \notin K \otimes \alpha$. Then by *extensionality* $(\alpha \rightarrow \beta) \notin K \otimes ((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta))$ and $(\alpha \rightarrow \neg\beta) \in K \otimes ((\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \neg\beta))$. Hence by $(C \leq)$, $(\alpha \rightarrow \neg\beta) \leq (\alpha \rightarrow \beta)$ and $(\alpha \rightarrow \beta)$ \not $<$ $(\alpha \rightarrow \neg\beta)$ and consequently $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$. For the other direction if $\alpha \vdash \bot$, then by *closure* and *success* it follows that $\beta \in K \otimes \alpha$ for all β. Let $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$. Then by $(C \leq)$ and *extensionality* $(\alpha \rightarrow \beta) \in K \otimes \alpha$; hence by *closure* and *success* $\beta \in K \otimes \alpha$. ■

Proof of Theorem 18

Due to proof of Theorem 16 we just add the proof of preservation.

Preservation: Let $\neg\alpha \notin K$. Let $\beta \in K$. Then by primacy $\neg\alpha < \beta$. By dominance $\beta \leq \alpha \rightarrow \beta$. Then by transitivity $\neg\alpha < \alpha \rightarrow \beta$. Hence by **Lemma 26** $\alpha \rightarrow \neg\beta < \alpha \rightarrow \beta$ from which we can conclude by (*EBR*) that $\beta \in K \otimes \alpha$. ■

Proof of Theorem 19

Due to proof of Theorem 17 we just need to prove primacy:

Primacy: Let $\alpha \notin K$ and $\beta \in K$. Then $\alpha \wedge \beta \notin K$. By *preservation* $K \subseteq K \otimes \neg(\alpha \wedge \beta)$. Then $\beta \in K \otimes \neg(\alpha \wedge \beta)$, from which it follows from (C≤) that $\alpha \leq \beta$. Due to $\alpha \wedge \beta \notin K$ it follows that $\nvdash \alpha \wedge \beta$, then by *consistency*, $\alpha \notin K \otimes \neg(\alpha \wedge \beta)$. Then by (C≤), $\beta \nleq \alpha$. Hence $\alpha < \beta$. ■

Proof of Theorem 20

The previous proofs of closure, success, consistency, extensionality and disjunctive factoring didn't use the fact that $\alpha \in K$ or $\alpha \notin K$. Then there are enough to proof the part of (EBR_2) in the case that $\alpha \notin K$. If $\alpha \in K$, closure, success, consistency, extensionality, and disjunctive factoring trivially follow. Vacuity 2 trivially follows from (EBR_2). ■

Proof of Theorem 21

We just need to prove (EBR_2):

(EBR_2) If $\alpha \in K$ it follows trivially by *vacuity 2*. If $\alpha \notin K$ the proof is equal to (EBR).■

References

[1] Carlos Alchourrón, "Philosophical foundations of deontic logic and the logic of defeasible conditionals", in J.J.Meyer and R.J.Wieringa, editors, *Deontic Logic in Computer Science: Normative System Specificaction*, pages 43-84. Wiley and Sons, Cichester, 1993.

[2] Carlos Alchourrón, "Defeasible Logic: Demarcation and Affinities", in G. Crosso, L. Fariña Del Cerro, and A. Herzig, editors, *Conditional and Artificial Intelligence*, Oxford University Press, 1994.

[3] Carlos Alchourrón, "Detachment and Defeasibility in Deontic Logic", *Studia Logica*, (51):5-18, 1996.

[4] Carlos Alchourrón, Peter Gärdenfors, and David Makinson, "On the Logic of Theory Change: Partial Meet Contraction and Revision Function", *Journal Symbolic Logic*, 50:510-530, 1985.

[5] Carlos Alchourrón and David Makinson, "Hierarchies of Regulations and their Logic", in Risto Hilpinen, editor, *New Studies in Deontic Logic: Norms, Actions, and the Foundations of Ethics*, pages 125-148, 1981.

[6] Lennart Åquist, "Modal Logic with Subjunctive Conditionals and Dispositional Predicates", *Journal of Philosophical Logic*, 2:1-76, 1973.

[7] Verónica Becher, *Binary Functions for Theory Change*, PhD thesis, Departamento de Computación, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, Septiembre 1999.

[8] Verónica Becher, Eduardo Fermé, Sandra Lazzer, Carlos Oller, Gladys Palau, and Ricardo Rodríguez, "Some Observations on Carlos Alchourrón's Theory of Defeasible Conditionals", in Mc. Namara P. and Prakke, editors, *Norms, Logics and Information Systems*, New Studies on Deontic Logic and Computer Science, pages 219-230. Ios Press, Amsterdam/Tokio/Washinton DC, 1999.

[9] Peter Gärdenfors, *Knowledge in Flux: Modeling the Dynamics of Epistemic States*, The MIT Press, Cambridge, 1988.
[10] Peter Gärdenfors and D. Makinson, *Revision of Knowledge Systems using Epistemical Entrenchment*, pages 83-95, Morgan Kaufmann, Los Altos, m. vardi edition, 1988.
[11] Peter Gärdenfors and David Makinson, "Relations Between the Logic of Theory Change and Nonmonotonic Logic", in A.Fuhrmann and M.Morreau, editors, *The Logic of Theory Change*, volume 465 of *Lecture Notes in Artificial Intelligence*, pages 185-205. Springer Verlag, Berlin, 1991.
[12] Peter Gärdenfors and David Makinson, "Nonmonotonic Inference Based on Expections", *Artificial Intelligence*, 65:197-245, 1994.
[13] Peter Gärdenfors, "Belief Revisions and the Ramsey Test for Conditionals", *Philosophical Review*, 95:81-93, 1986.
[14] Adam Grove, "Two Modellings for Theory Change", *Journal of Philosophical Logic*, 17:157-170, 1988.
[15] Sven Ove Hansson, *A Textbook of Belief Dynamics. Theory Change and Database Updating,* Applied Logic Series, Kluwer Academic Publishers, Dordrecht, 1999.
[16] Sten Lindström and Wlodek Rabinowicz, "Epistemic Entrenchment with Incomparabilities and Relational Belief Revision", in A. Fuhrmann and M. Morreau, editors, *The Logic of Theory Change*, Volume 465 of *Lecture Notes in Artificial Intelligence*, pages 93-126. Springer Verlag, Berlin, 1991.
[17] Maurice Pagnucco, *The Role of Abductive Reasoning Within the Process of Belief Revision*, PhD thesis, Department of Computer Science, University of Sydney, February 1996.
[18] Maurice Pagnucco, Abhaya Nayak, and Norman Foo, "Abductive Expansion: Abductive Inference and the Process of Belief Change", in C. Zhang, J. Debenham, and D. Lukose, editors, *Proceedings of the Seventh Australian Joint Conference on Artificial Intelligence (AI94)*, pages 267-274, Armidale, Australia, 1994.
[19] Hans Rott, "Conditionals and Theory Change: Revision, Expansions, and Additions", *Synthese*, 81:91-113, 1989.
[20] Hans Rott, "A Nonmonotonic Conditional Logic for Belief Revision. Part 1: Semantics and Logic of Simple Conditionals", in A. Fuhrmann and M. Morreau, editors, *The Logic of Theory Change*, Volume 465 of *Lecture Notes in Artificial Intelligence*, pages 93-126, Springer Verlag, Berlin, 1991.
[21] Hans Rott, "Two Dogmas of Belief Revision", *Journal of Philosophy*, 97(9):503-522, 2000.