TENSORIAL EQUATIONS FOR THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER FLOWS

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Abstract— **Tensorial equations are derived for a laminar and attached boundary layer flow with null pressure gradient in the normal direction to a smooth three-dimensional surface. An incompressible, isothermal and viscous fluid of Newtonian type is assumed. Covariant derivatives in the three dimensional Euclidean domain are employed, where the surface curvature terms are implicitly included in the Christoffel symbols with the aim of writing the boundary layer equations in an invariant form irrespective to the particular choice of the coordinate system. These equations are covariant under a linear coordinate transformation on the two surface coordinates, and a scaling along the normal direction to the surface. As a test case, the boundary layer near a sphere in an axisymmetrical steady flow is numerically computed using a pseudo-spectral approach.**

Keywords— **boundary-layer equations, laminar steady flow, incompressible viscous fluid, three-dimensional surfaces, tensor analysis.**

I. INTRODUCTION

As it is well known, mechanical devices may require the calculation of fully three-dimensional boundary layers (e.g. see Schlitchting and Gersten, 2004), as those associated with flow inside turbomachines (Lakshminarayana, 1995), horizontal-axis wind turbine blades (Prado, 1995), laminar flow technology (Stock, 2006) or aerospace technology (Dwoyer *et al.*, 1978), among other cases. The laminar boundary layer equations for three-dimensional surfaces are well-known (e.g. see Cebeci and Cousteix, 1999; Dey, 2001). In any case, there are curvature effects that do not disappear as they do in the two-dimensional case (Reed and Lin, 1993). In fact, these curvature effects are present through the (local) principal curvatures of the surface. If one of them is null the boundary layer equations are independent of the curvature effects, as in planes, yawed infinite cylinders or wings (Pai, 1956).

There is nothing special about a particular system of coordinates, so a physical or geometrical law behind the equations for the boundary layer flow should be independent of them. The related equation is often first presented in Cartesian coordinates, and its differential form is typically presented in a coordinate independent system using vector notation based on the nabla operator (German, 2007). With the coordinate invariant form known, the solution is usually approached by selecting a particular coordinate system, e.g. Cartesian, cylindrical, or spherical ones. In order to determine a differential equation in a selected coordinate system, the nabla operator is expressed in terms of partial derivatives with respect to the chosen coordinates, or alternatively, the related operators are obtained through a tedious coordinate transformation from the well known Cartesian form. In this way, coordinate independent expressions of these equations can be cast in a specific form from which analytical and numerical solutions can be pursued. An alternative although equivalent approach to posing coordinate invariant equations is to employ a tensorial representation. As it is well known, a tensor equation is a coordinate invariant equation where the monomials have to be tensors of a same order and, in component format, all terms must contain the same free indices. Thus, tensorial equations, like those written in terms of the nabla operator, are form invariant, or covariant, with respect to regular curvilinear coordinate transformations. As in the case with the nabla form, operations such as the gradient and divergence are well defined. However their representation is in terms of the resource of tensor calculus. Besides the fact that both approaches are equivalents, the tensor approach often provides a more tractable and compact method for dealing with transformations among regular coordinate systems, since the relationships between the differential equation and the geometry can be a bit more clean than the nabla operator approach. For instance, a differential equation can be cast in terms of a particular set of coordinates through the specification of the metric tensor corresponding to the coordinate system, where all of the differential operators depend exclusively on metric tensor and combinations of its partial derivatives, i.e. the Christoffel symbols. Although determining the metric tensor and Christoffel symbols for a regular but arbitrary coordinate system is not a simple task, it is often appreciably less challenging and tedious than determining the gradient and divergence operators by direct transformation of the partial derivatives. This is especially true for cases when these operators are not tabulated for the desired coordinate system, whereas the tensor components and Christoffel symbols may also not be tabulated. Nevertheless, a straightforward recipe is available through which they can be calculated with tensor analysis (German, 2007).

In another way, Panaras (1987) gave the formulation of the unsteady, compressible Navier–Stokes covariant equations in general non-orthogonal curvilinear coordinates and thereafter a discussion follows about the terms which could be omitted in a thin shear-layer formulation. The present work considers steady, incompressible fluid flow using covariant derivative in an orthogonal coordinate system but it is not a special case. The main differences are: (i) Panaras wrote the boundary layer equations in a full way with several surface curvature terms explicitly given, while in this work, they are implicitly included in the Christoffel symbols of the covariant derivatives with the main aim of putting the boundary layer equations in invariant form irrespective of the choice of the coordinate system, and (ii) a slight difference exists in the choice of the normal coordinate to the surface. As it is well known in standard boundary-layer engineering, the surface curvature terms (i.e. the normal pressure gradient) are the first terms to be added when Prandtl basic equations are extended to the three dimensional case and there are many expressions proposed in literature, partially due to the several approximations that can be performed with the curvature terms. However, it is cumbersome to check about the physical invariance of the proposed equations. Therefore, in order to obtain a more physical picture and an easier verification, it would be convenient to place the surface curvature terms only in the covariant derivatives instead of displaying them in boundary layer equations.

On the other hand, a full numerical solution of the Navier–Stokes equations for high Reynolds numbers is nowadays a standard approach, like those based in the Reynolds Averaged Navier–Stokes (RANS) equations (Wilcox, 1998), Large Eddy Simulations (LES) (Sagaut, 2001) and Direct Numerical Simulations (DNS) (Pope, 2003), among several approaches. Nevertheless, three-dimensional boundary layers and boundary layer separation are still areas where boundary layer computations may be seen as a complementary tool, for instance, coupled viscous-inviscid computations for wind turbine airfoil flows (Bermúdez *et al.*, 2002), unsteady flows in turbomachinery (Epureanu *et al.*, 2001), unsteady interaction with outer transonic flows (Bogdanov and Diyesperov, 2005), flows past moving bands (Frondelius *et al.*, 2006), as well as oscillatory boundary layers attached to deformable solid walls (Nicols and Vega, 2003).

The boundary layer equations are also proposed as alternatives to the classical convective outlet boundary condition heavily used for wall-bounded laminar/turbulent flows (Fournier *et al.*, 2008). In fact, three-dimensional boundary layer methods are likely to be both more accurate and less computationally expensive than a full numerical solution of the Navier–Stokes equations. They can be performed using classical approaches, as the integral ones used in Karimipanah and Olsson (1993), where the effects of rotation and compressibility on rotor blade boundary layers are studied. Other methods are finite differences (Anderson, 1985), finite elements (Schetz, 1991) and pseudo-spectral approaches (Storti, 1998). In any case, the solutions can be used for testing more elaborated computational codes as those that directly solve the Navier–Stokes equations in several contexts, for instance, for modelling free surface flows (D'Elía *et al.*, 2000; Battaglia *et al.*, 2006; Storti *et al.*, 1998), added mass computations (Storti and D'Elía, 2004) or inertial waves in closed flow domains (D'Elía *et al.*, 2006).

The present work develops a covariant derivation of the boundary layer equations in tensorial form valid for three-dimensional, steady and laminar boundary layers with null-pressure gradient in the normal direction to the surface. They are written in an orthogonal curvilinear coordinate system, defined by two surface coordinates plus a normal one to the surface, such as, (i) they are covariant under a coordinate transformation on the two intrinsic surface coordinates and a scaling one along the normal coordinate to the body surface; (ii) the surface curvature terms are implicitly present in the covariant derivatives and not in the displayed equations as opposed to the more traditional way that typically includes the surface curvature, surface principal curvatures or related terms in the boundary layer equations; (iii) it only remains the surface metric tensor in the continuity equation. As a practical use of the proposed covariant formalism, the steady and laminar three-dimensional boundary layer flow near a moving sphere of an incompressible and viscous fluid is performed as a test case using a pseudo-spectral approach, first without a rotation and later with a steady spin rotation.

II. SURFACE AND EXTENDED COORDINATES

In all this work, Latin affixes run in the three-dimensional Cartesian space, e.g. $i = 1, 2, 3$, the Greek ones run on the two-dimensional surface, e.g. $\alpha = 1, 2$, while upper and lower affixes denote contravariant and covariant components of tensors of any range (Aris, 1989; Smith, 1963). Lower and upper letters denote

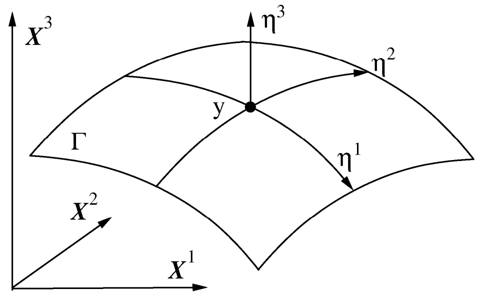

Figure 1: Dimensionless coordinates: surface intrinsic (η^1, η^2) on the smooth surface Γ, the normal coordinate η^3 and the point $\boldsymbol{y} = (y^1, y^2, y^3) \in \Gamma$ in the three-dimensional Cartesian space $\mathbb{R}^3$.

dimensionless and dimension quantities, respectively. The basic ones are the coordinates $x^i = X^i/L$, velocities $u^i = U^i/U$ and pressure $p = P/(\rho U^2)$, where U, L and ρ are typical speed, length and fluid density, respectively, while, regarding incompressibility of the fluid, ρ is constant.

It is assumed that the body surface Γ is smooth enough to be represented by a coordinate grid on the surface itself, expressing the position of a surface point in terms of the dimensionless surface coordinates. Thus, the surface equation has the form $x^i = x^i(\eta^1, \eta^2)$, where η^1, η^2 are the surface coordinates, the intrinsic ones, while the lines $\eta^1 =$ constant and $\eta^2 =$ constant are the coordinate lines. The Cartesian coordinates $\boldsymbol{y} = (y^1, y^2, y^3)^T \in \mathbb{R}^{3\times 1}$, for a generic point $\boldsymbol{y}$ on the surface Γ can be expressed as a function of the surface coordinates $\boldsymbol{y} = \boldsymbol{y}(\eta^1, \eta^2)$, see Fig. 1, where $(...)^T$ denotes the transpose. An extended coordinate system (η^1, η^2, η^3) is introduced as a three-dimensional curvilinear one for extending the surface coordinates to the three-dimensional space where the body is immersed, where the η^3-coordinate is normal to the body surface. When the generic point $\boldsymbol{y}$ is over the surface, the coordinates are $(\eta^1, \eta^2, 0)$, and when it is at any place in the three-dimensional space, the coordinates are $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{n}\delta$, where $\boldsymbol{n} = \boldsymbol{n}(\eta^1, \eta^2)$ is the exterior unit normal on the surface at point $\boldsymbol{y} = \boldsymbol{y}(\eta^1, \eta^2)$, with $\boldsymbol{y} \in \Gamma$, while $\delta = \delta(\eta^1, \eta^2)$ is a dimensionless boundary layer thickness (or expansion parameter), with $0 < \delta \ll 1$. The Jacobian of the transformation from the (x^1, x^2, x^3) system to the (η^1, η^2, η^3) one is assumed as regular in all the domain and it is given by $J_{ij} = \partial x_i/\partial \eta_j$, for $1 \le i, j \le 3$, which are written as the matrix $\boldsymbol{J} = [\boldsymbol{J}_1\ \boldsymbol{J}_2\ \boldsymbol{J}_3] \in \mathbb{R}^{3\times 3}$, whose columns are

$$\begin{aligned} \boldsymbol{J}_1 &= \frac{\partial \boldsymbol{x}}{\partial \eta^1} = \frac{\partial \boldsymbol{y}}{\partial \eta^1} + \eta^3 \frac{\partial \delta}{\partial \eta^1}\boldsymbol{n} + \eta^3 \delta \frac{\partial \boldsymbol{n}}{\partial \eta^1} \ ; \\ \boldsymbol{J}_2 &= \frac{\partial \boldsymbol{x}}{\partial \eta^2} = \frac{\partial \boldsymbol{y}}{\partial \eta^2} + \eta^3 \frac{\partial \delta}{\partial \eta^2}\boldsymbol{n} + \eta^3 \delta \frac{\partial \boldsymbol{n}}{\partial \eta^2} \ ; \\ \boldsymbol{J}_3 &= \frac{\partial \boldsymbol{x}}{\partial \eta^3} = \boldsymbol{n}\ \delta \ . \end{aligned} \tag{1}$$

III. SPATIAL AND SURFACE METRIC TENSORS

The metric tensor is an intrinsic quantity that relates measurements performed inside a same domain, spatial or surface one. Note that in this work there is a spatial metric tensor g_{ij} for the three-dimensional domain, with $1 \le i, j \le 3$, as well as a surface metric tensor $a_{\alpha\beta}$ for the body surface, with $1 \le \alpha, \beta \le 2$. The main attention in this work it is focused on the former one. Both of them are assumed as smooth, symmetric and positive definite tensors.

A. Partition of the spatial metric tensor

The spatial metric tensor can be expressed in covariant g_{ij} or in contravariant g^{ij} components, taking in the first case the form

$$g_{ij} = \sum_{h=1}^{3} \frac{\partial x^h}{\partial \eta^i}\frac{\partial x^h}{\partial \eta^j} = \boldsymbol{x}_{,i}^T \cdot \boldsymbol{x}_{,j} \ ; \tag{2}$$

where $\boldsymbol{x}_{,i} = \partial \boldsymbol{x}/\partial \eta^i$. The tensor g_{ij} can be split in surface $g_{\alpha\beta}$, normal g_{33}, and mixed $g_{\alpha 3}$, $g_{3\alpha}$ tensor components. The corresponding covariant and contravariant matrices are $\boldsymbol{g}_{ij} \equiv [g_{ij}]$ and, $\boldsymbol{g}^{ij} \equiv [g^{ij}]$, respectively, and they are related by $\boldsymbol{g}^{ij} = (g_{ij})^{-1}$, where $(...)^{-1}$ denotes the inverse. The metric tensor partition suggests the corresponding matrix one

$$\boldsymbol{g}_{ij} = \begin{bmatrix} g_{\alpha\beta} & g_{\alpha 3} \\ g_{3\beta} & g_{33} \end{bmatrix} \ ; \tag{3}$$

where $1 \le i, j \le 3$ for the extended coordinates, and $1 \le \alpha, \beta \le 2$ for the surface coordinates, with $g_{\alpha\beta} \in \mathbb{R}^{2\times 2}$, $g_{\alpha 3} \in \mathbb{R}^{2\times 1}$, $g_{3\beta} \in \mathbb{R}^{1\times 2}$ and $g_{33} \in \mathbb{R}$.

B. Asymptotic orders in the covariant and contravariant spatial metric tensor

The asymptotic orders in the covariant g_{ij} and contravariant g^{ij} spatial metric tensor with respect to the expansion parameter δ, with $0 < \delta \ll 1$, are inferred in Appendices A. and B., and they are summarized in Table 1, where $a_{\alpha\beta}$ is assumed as a regular tensor.

Spatial metric tensor			
covariant g_{ij}		contravariant g^{ij}	
$g_{\alpha\beta}$	$a_{\alpha\beta} + O(\delta)$	$g^{\alpha\beta}$	$a_{\alpha\beta}^{-1} + O(\delta)$
$g_{\alpha 3}$	$O(\delta^2)$	$g^{\alpha 3}$	$O(1)$
g_{33}	δ^2	g^{33}	$\delta^{-2} + O(\delta^{-1})$

Table 1: Covariant g_{ij} and contravariant g^{ij} spatial metric tensor components as a function of the expansion parameter δ, with $0 < \delta \ll 1$.

C. Asymptotic orders in the Christoffel symbols

It is necessary to consider the Christoffel symbols since they appear in the expressions for the covariant derivatives. As it is well known, they have not got a tensor character, that is, they are not transformed with a tensorial law under a linear coordinates transformation (Smith, 1963), and they can be calculated with

$$[ij,k] \equiv \begin{bmatrix} i \; j \\ k \end{bmatrix} = \frac{1}{2}\left[\frac{\partial g_{ik}}{\partial \eta^j} + \frac{\partial g_{jk}}{\partial \eta^i} - \frac{\partial g_{ij}}{\partial \eta^k}\right] \quad ; \qquad (4)$$

for the first kind and

$$\mathit{\Gamma}_{ij}^k \equiv \{i, jk\} \equiv \left\{\begin{matrix} i \\ j \; k \end{matrix}\right\} = g^{ih}[jk,h] \; ; \qquad (5)$$

for the second kind. It is noted that, as the metric tensor g_{ij} is assumed as a symmetric one, these symbols are also symmetric in the lower affixes and consequently the torsion tensor $T_{ij}^k = \mathit{\Gamma}_{ij}^k - \mathit{\Gamma}_{ji}^k$ is null, that is, the spatial and surface domains are torsion-free (Morgan, 1993). There are six types of these symbols depending on whether the affixes i, j, k are on the smooth surface (η^1 or η^2) or on the normal η^3 coordinate. The asymptotic order of the Christoffel symbols with respect to the expansion parameter δ, with $0 < \delta \ll 1$, are estimated from the order of the spatial metric tensor in Appendices E. and F., and they are summarized in Table 2. Note that there is only one special case given by the symbol $\{3, \beta\ \gamma\} = O(\delta^{-1})$, while the remaining ones are $O(\delta)$ or $O(\delta^2)$.

Christoffel symbols			
1st kind		2nd kind	
term	order	term	order
$[\alpha\beta, \gamma]$	$O(1)$	$\{\alpha, \beta\gamma\}$	$O(1)$
$[\alpha\beta, 3]$	$O(\delta)$	$\{3, \beta\gamma\}$	$O(\delta^{-1})$
$[3\beta, \gamma]$	$O(\delta)$	$\{\beta, \alpha 3\}$	$O(\delta)$
$[3\beta, \gamma]$	$O(\delta^2)$	$\{3, \alpha 3\}$	$O(\delta)$
$[33, \gamma]$	$O(\delta^2)$	$\{\alpha, 33\}$	$O(\delta^2)$
$[33, 3]$	0	$\{3, 33\}$	$O(\delta^2)$

Table 2: Order of the Christoffel symbols of first and second kind as a function of the expansion parameter δ, with $0 < \delta \ll 1$.

IV. NAVIER–STOKES EQUATIONS IN CURVILINEAR COORDINATES

The dimensionless Navier-Stokes equations for a laminar, isothermal and steady flow without body forces of an incompressible and viscous fluid of Newtonian type with constant physical properties, are written in the extended coordinate system (η^1, η^2, η^3) as

$$\begin{aligned} u^i u_{,i}^k + g^{kj} p_{,j} &= Re^{-1} g^{ij} u_{,ij}^k \; ; \\ u_{,k}^k &= 0 \; ; \end{aligned} \qquad (6)$$

see for example Aris (1989), Sec. 8.22, and Spurk (1997), Sec. 4.1.3, which are the momentum and the continuity equations, respectively, u^k is the k-component of the flow velocity expressed in contravariant components, while $Re = UL/\nu$ is the Reynolds number, where ν is the fluid kinematic viscosity. It should be noted that as the pressure p is a scalar field, then, its gradient can be written as $p_{,j} = \partial p / \partial \eta^j$. For further use, the first and second covariant derivatives of a smooth vector field v^k are respectively computed with (see for example Smith (1963))

$$\begin{aligned} v_{,i}^k &\equiv \frac{\partial v^k}{\partial \eta^i} + \left\{\begin{matrix} k \\ h \; i \end{matrix}\right\} v^k \quad ; \\ v_{,ij}^k &\equiv \frac{\partial v_{,i}^k}{\partial \eta^j} + \left\{\begin{matrix} k \\ h \; j \end{matrix}\right\} v_{,i}^h - \left\{\begin{matrix} h \\ i \; j \end{matrix}\right\} v_{,h}^k \quad . \end{aligned} \qquad (7)$$

A. Tangential component of the momentum equations

The tangential components of the momentum equations to the wall surface, are extracted from (6) as

$$u^i u_{,i}^\alpha + g^{\alpha j} p_{,j} = Re^{-1} g^{ij} u_{,ij}^\alpha \quad . \qquad (8)$$

In Appendix C. it is shown that the convective, reduced pressure and viscous terms in Eq. (8) have the asymptotic orders

$$\begin{aligned} u^i u_{,i}^\alpha &= u^3 \frac{\partial u^\alpha}{\partial \eta^3} + u^\beta u_{,\beta}^\alpha + O(\delta) \quad ; \\ g^{\alpha j} p_{,j} &= g^{\alpha\beta} p_{,\beta} + O(\delta) \quad ; \\ Re^{-1} g^{ij} u_{,ij}^\alpha &= \frac{Re^{-1}}{\delta^2} \frac{\partial^2 u^\alpha}{\partial \eta^3 \partial \eta^3} + O(\delta) \quad . \end{aligned} \qquad (9)$$

Retaining the leading terms in Eq. (9), the tangential components of the momentum equations are reduced to

$$u^3 \frac{\partial u^\alpha}{\partial \eta^3} + u^\beta u_{,\beta}^\alpha + g^{\alpha\beta} p_{,\beta} = \frac{Re^{-1}}{\delta^2} \frac{\partial^2 u^\alpha}{\partial \eta^3 \partial \eta^3} \; . \qquad (10)$$

B. Normal component of the momentum equations

The normal components of the momentum equations to the wall surface are taken from (6) as

$$u^i u_{,i}^3 + g^{3j} p_{,j} = Re^{-1} g^{ij} u_{,ij}^3 \quad . \qquad (11)$$

In Appendix D. it is shown that the convective, reduced pressure and viscous terms in Eq. (11) give the asymptotic orders

$$\begin{aligned} u^i u_{,i}^3 &= O(\delta^{-1}) \; ; \\ g^{3j} p_{,j} &= O(\delta^{-1}) \; ; \\ Re^{-1} g^{33} u_{,33}^3 &= O(\delta^{-1}) \; ; \end{aligned} \qquad (12)$$

and

$$\begin{aligned} u^i u_{,i}^3 &= \left\{\begin{matrix} 3 \\ \alpha\beta \end{matrix}\right\} u^\alpha u^\beta + O(1) \; ; \\ g^{3j} p_{,j} &= \frac{1}{\delta^2} \frac{\partial p}{\partial \eta^3} + O(1) \; ; \end{aligned} \qquad (13)$$

respectively. Comparing Eqs. (12,13) it follows that

$$\frac{\partial p}{\partial \eta^3} = O(\delta) \; ; \qquad (14)$$

which is equivalent to the well known assumption in boundary layer analysis at zero order that the pressure gradient is negligible along the direction normal to the wall.

C. Continuity equation

The divergence of the velocity u^k can be calculated (e.g. see Aris (1989), Sec. 7.56) with

$$u^k_{,k} = \frac{1}{\sqrt{g^0}} \frac{\partial}{\partial x^k} \left(\sqrt{g^0} u^k \right) \; ; \tag{15}$$

since $g^0 = a^0 \delta^{-2} + O(1)$, where $g^0 = \det(g^{ij})$ and $a^0 = \det(a^{\alpha\beta})$ are the determinant of the spatial and surface metric tensors, respectively. The equation (15) can be split and rewritten as

$$\frac{\partial u^3}{\partial \eta^3} + \frac{1}{\delta} \, (\delta \, u^\alpha)_{,\alpha} = 0 \quad . \tag{16}$$

D. Tensorial equations for a three dimensional laminar boundary layer flow

Finally, dimensionless covariant equations for a steady, laminar, attached and three dimensional boundary layer flow with null pressure gradient in the normal direction to the body surface are given by collecting Eqs. (10, 14, 16),

$$\begin{aligned} u^3 \frac{\partial u^\alpha}{\partial \eta^3} + u^\beta u^\alpha_{,\beta} + g^{\alpha\beta} p_{,\beta} &= \frac{Re^{-1}}{\delta^2} \frac{\partial^2 u^\alpha}{\partial \eta^3 \partial \eta^3} \; ; \\ \frac{\partial p}{\partial \eta^3} &= 0 \; ; \\ \frac{\partial u^3}{\partial \eta^3} + \frac{1}{\delta} \, (\delta \, u^\alpha)_{,\alpha} &= 0 \quad . \end{aligned} \tag{17}$$

In practical computations, it could be more convenient to put the pressure gradient $p_{,\alpha} = \partial p / \partial \alpha$ in the form $g^{\beta\alpha} p_{,\alpha} = v^\alpha v^\beta_{,\alpha}$, where v are the contravariant components of the outer (inviscid) velocity field, e.g. see Storti (1998).

V. NUMERICAL EXAMPLE

As a single application of the covariant three dimensional boundary layer equations given by Eq. (17), an axially symmetric flow past a sphere of radius A is numerically solved using a pseudo-spectral like approach. The numerical method is based on Fourier expansion in the lateral transformed coordinate, similar to the transformation that leads to the polynomial Tchebischev expansion in finite intervals, although more appropriate to semi-infinite intervals in such a way that no extra parameter is needed for the outer boundary of the layer. A scaling is applied to the coordinate normal to the body surface with an innovation based on the computed boundary layer thickness without assuming a priori variation for it. Thorough details about this computation and other flow tests including cones, yawed flat plates and circular cylinders were given in Storti (1998) so that only a brief account is given here.

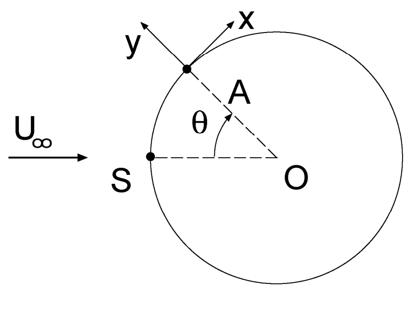

Figure 2: Curvilinear coordinates Y^1, Y^3 and polar angle $\theta = Y^1/A$, measured from the forward stagnation point S on a sphere of radius A, the generic point on the body surface is $P(Y^1, 0)$.

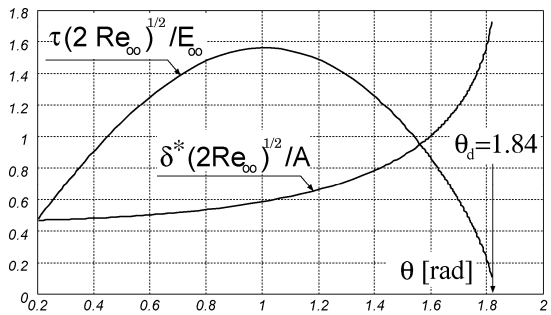

Figure 3: Non-dimensional wall friction $\tau\sqrt{2Re_\infty}/E_\infty$ and displacement thickness $\delta^*\sqrt{2Re_\infty}/A$, as a function of θ using the potential velocity profile. The separation point is at $\theta_d \approx 1.84$ radians ($\approx 105.45°$).

The sphere is immersed in an incompressible and viscous fluid of Newtonian type, with kinematic fluid viscosity ν and fuid density ρ. Two flow cases are considered.

The first flow case is a sphere moving with constant velocity U_∞ and no rotation, where two approximations performed for the base flow on the body surface $\boldsymbol{V} = (V^1, 0, 0)$, with $V^1 = V^1(\theta)$, where $\theta = Y^1/A$ is the polar angle from the forward stagnation point S, and Y^1 is the curvilinear coordinate along the wall measured from the same point, see figure 2.

On one hand, the first approximation assumes the potential velocity profile $V^1 = (3/2)\; U_\infty \sin\theta$, with $0 \leq \theta \leq \pi$. The non-dimensional wall friction $\tau\sqrt{2Re_\infty}/E_\infty$ and the displacement thickness $\delta^*\sqrt{2Re_\infty}/A$ are obtained solving (17) and they are shown in figure 3 as a function of the polar angle θ, where $E_\infty = \rho U_\infty^2/2$ and the exterior Reynolds number $Re_\infty = U_\infty D/\nu$ are computed using the sphere diameter $D = 2A$, with wall friction $\tau = \nu(\partial U^1/\partial Y^3)|_{Y^3=0}$, where Y^3 is the normal coordinate,

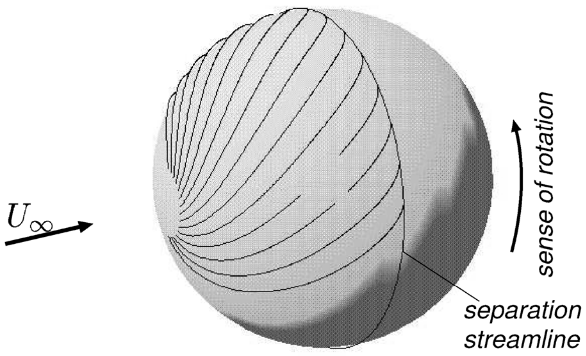

Figure 4: Computed limit viscous streamlines for the sphere of radius A, moving with a steady spin rotation ω around an axis parallel to the constant velocity U_∞, when $\omega A/U_\infty = 1$.

and displacement thickness δ^* is computed from

$$U_\infty \delta^* = \int_0^\infty (U_\infty - u) dy \; . \tag{18}$$

The separation point is found at $\theta_d \approx 1.84$ radians ($\approx 105.45°$), in good agreement with the separation predictions given by a Rott-Cratree semi-analytical and a finite-difference computations, between 103.6° and 105.5°, respectively, as cited by White (2005).

On the other hand, for super-critical Reynolds numbers, the experimental velocity profile does not differ too much from the potential flow model altough there are flow separation effects. Since the axially symmetrical Blasius profile series is known up to the term θ^7, the function $\sin\theta$ is replaced by a series in odd powers giving a more realistic velocity profile. Thus, the actual velocity profile measured by Fage (1936) and cited by White (2005) at $Re = 200\ 000$ fits the curve

$$\frac{U}{U_\infty} = \frac{3}{2}\,\theta - 0.4371\,\theta^3 + 0.1481\,\theta^5 - 0.0423\,\theta^7 \quad ; \tag{19}$$

for $0 \leq \theta \leq 1.48$, which drops off much faster than the potential one, reaching a maximum of $U/U_\infty = 1.274$ at $\theta = 1.291$ radians ($\approx 74°$) while the potential velocity profile has a maximum of $U/U_\infty = 3/2$ at $\theta = 1.571$ radians ($\approx 90°$). The flow separation is reached at $\theta''_d = 1.424$ radians ($\approx 81.6°$), which is again in good agreement with the predictions given by a Rott-Cratree semi-analytical and a finite-difference computation, between 81.1° and 82.4°, respectively.

In the second flow case, the velocity profile given by (19) is again assumed, but the sphere has a steady spin $\boldsymbol{\omega}$ around an axis parallel to the constant velocity $\boldsymbol{U}_\infty$, such as $\omega A/U_\infty = 1$. It is verified that whereas the inviscid streamlines are simply meridians, the limit viscous streamlines in this case have a tendency to rotate with the sphere, until they align with the separation streamline that is a parallel at $\theta'''_d = 1.47$ radians ($\approx 84.2°$) from the forward stagnation point, as it is shown in figure 4. Note that the spinning tends to stabilize the boundary layer against separation, resulting in a delay of almost 0.052 radians ($\approx 3°$) with respect to $\theta''_d = 1.424$ radians ($\approx 81.6°$). This is due to the centrifugal force that can be assimilated to a pressure gradient directed to the equator $\theta \to \pi/2$ radians ($\theta \to 90°$). Since the boundary layer separation in this case happens before the equator, this is equivalent to a favorable pressure gradient which has also a significant incidence in the viscous drag.

VI. CONCLUSIONS

An attached, laminar, isothermal steady flow without body forces of an incompressible viscous fluid of Newtonian type and constant physical properties has been considered. The boundary layer equations were first written in orthogonal extended curvilinear three-dimensional coordinates η^i, with $i = 1, 2, 3$. Next, the approximation for the two-dimensional surface coordinates η^α, with $\alpha = 1, 2$, were derived from the asymptotic properties of the metric tensor g^{ij} with respect to the dimensionless boundary layer thickness $\delta(\eta^1, \eta^2)$, with $0 < \delta \ll 1$. Six types of Christoffel symbols were found, depending on whether the affixes i, j, k are over the smooth surface, η^1 and η^2, or along the normal direction η^3. The order of these symbols with respect to the expansion parameter δ were estimated from the corresponding ones of the metric tensor, which are summarized in table 2. The well known consequence of zero pressure variation across the boundary layer is recovered in the continuity equation (second line in (17)). From a practical point of view, some advantages of writing the boundary layer equations in a covariant formalism in the three-dimensional Euclidean space are:

1. It is more easy to check the physical units of each term and to detect errors due to mistakes in the approximations performed in the derivations;
2. Any intrinsic curvature term is placed only in the covariant derivatives rather than displayed in the boundary layer equations in order to obtain a more clean physical picture and easier verification;
3. There is not curvature terms in the boundary layer equations and it only remains the surface metric tensor a^{ij} in the continuity equation;
4. It is a bit easier to change from one coordinate system to another in a three-dimensional Euclidean space, since there are not complicated curvature terms related to the non-Euclidean nature of the two-dimensional surface;
5. The equations are covariant under a linear coordinate transformation on the two intrinsic surface coordinates η^1, η^2, and an arbitrary scaling of the normal coordinate η^3 to the body surface;
6. Finally, the covariant form also shows the well known properties of the boundary layer equations: a diffusive character on the normal co-

ordinate η^3 and a wave-like one on the surface coordinates η^1, η^2, that is, it has a dual intrinsic parabolic–hyperbolic nature.

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APPENDIX

A. Asymptotic orders in the covariant spatial metric tensor

For a point $\boldsymbol{y} \in \Gamma$, the partial derivatives $\boldsymbol{y}_{,1}$, $\boldsymbol{y}_{,2}$ are parallel to the surface Γ and, in the same way, the partial derivatives $\boldsymbol{n}_{,1}$ and $\boldsymbol{n}_{,2}$ are also parallel to the surface Γ. Therefore, their scalar product with respect to the unit normal to the body surface $\boldsymbol{n}$ are nulls, i.e. $\langle \partial \boldsymbol{y}^T / \partial \eta^\alpha, \boldsymbol{n} \rangle = 0$ and $\langle \partial \boldsymbol{n}^T / \partial \eta^\alpha, \boldsymbol{n} \rangle = 0$. There are three types of components in the covariant metric tensor g_{ij}: surface, mixed and normal components. First, the surface components $g_{\alpha\beta}$ are written as

$$
\begin{aligned}
g_{\alpha\beta} &= \frac{\partial \boldsymbol{x}^T}{\partial \eta^\alpha} \cdot \frac{\partial \boldsymbol{x}}{\partial \eta^\beta} \\
&= \frac{\partial \boldsymbol{y}^T}{\partial \eta^\alpha} \cdot \frac{\partial \boldsymbol{y}}{\partial \eta^\beta} + 2\eta^3 \frac{\partial \boldsymbol{y}^T}{\partial \eta^\alpha} \cdot \frac{\partial \boldsymbol{n}}{\partial \eta^\beta}\, \delta \\
&+ \left(\eta^3\right)^2 \frac{\partial \boldsymbol{n}^T}{\partial \eta^\alpha} \cdot \frac{\partial \boldsymbol{n}}{\partial \eta^\beta}\, \delta^2 + \left(\eta^3\right)^2 \frac{\partial \delta}{\partial \eta^\alpha} \frac{\partial \delta}{\partial \eta^\beta} \\
&= a_{\alpha\beta} + 2\eta^3 b_{\alpha\beta}\, \delta + \left(\eta^3\right)^2 c_{\alpha\beta}\, \delta^2 \\
&+ \left(\eta^3\right)^2 d_{\alpha\beta} = a_{\alpha\beta} + O(\delta) \; ;
\end{aligned} \qquad \text{(A1)}
$$

where

$$
\begin{bmatrix} a_{\alpha\beta} & b_{\alpha\beta} \\ c_{\alpha\beta} & d_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} \langle \boldsymbol{y}^T_{,\alpha}, \boldsymbol{y}_{,\beta} \rangle & \langle \boldsymbol{y}^T_{,\alpha}, \boldsymbol{n}_{,\beta} \rangle \\ \langle \boldsymbol{n}^T_{,\alpha}, \boldsymbol{n}_{,\beta} \rangle & \delta_{,\alpha} \delta_{,\beta} \end{bmatrix} \qquad \text{(A2)}
$$

which are functions of the surface coordinates η^1, η^2. The first term $a_{\alpha\beta}$ gives the intrinsic components of the surface metric tensor. Note from Eq. (A1) that near the surface the components of the metric tensor $g_{\alpha\beta}$ are equal, at first order, to the intrinsic $a_{\alpha\beta}$ ones. Next, the mixed components $g_{\alpha 3}$ are written as

$$
g_{\alpha 3} = \frac{\partial \boldsymbol{x}^T}{\partial \eta^\alpha} \cdot \frac{\partial \boldsymbol{x}}{\partial \eta^3} = \eta^3 \frac{\partial \delta}{\partial \eta^\alpha}\, \delta = O(\delta^2) \; . \qquad \text{(A3)}
$$

Finally, the normal component g_{33} is reduced to

$$
g_{33} = \frac{\partial \boldsymbol{x}^T}{\partial \eta^3} \cdot \frac{\partial \boldsymbol{x}}{\partial \eta^3} = \boldsymbol{n}\delta \cdot \boldsymbol{n}\delta = \delta^2 \; . \qquad \text{(A4)}
$$

Then, using the asymptotic orders given by Eqs. (A1,A3,A4), the covariant metric tensor, expressed in matrix notation and partitioned as in Eq. (3), has the asymptotic expansion

$$
\boldsymbol{g}_{ij} = [g_{ij}] = \begin{bmatrix} a_{\alpha\beta} + O(\delta) & O(\delta^2) \\ O(\delta^2) & \delta^2 \end{bmatrix} = \boldsymbol{STS} \; ; \quad \text{(A5)}
$$

where $\boldsymbol{a}_{\alpha\beta} = [a_{\alpha\beta}] \in \mathbb{R}^{2\times 2}$, while $\boldsymbol{T}, \boldsymbol{S} \in \mathbb{R}^{3\times 3}$, with

$$
\boldsymbol{T} = \begin{bmatrix} a_{\alpha\beta} + O(\delta) & O(\delta) \\ O(\delta) & 1 \end{bmatrix} \; ; \qquad \text{(A6)}
$$

and

$$
\boldsymbol{S} = \text{diag}\{1, 1, \delta\} \; ; \qquad \text{(A7)}
$$

where $\boldsymbol{S}$ is a regular matrix whether $\delta > 0$. Then, the first column of Table 1 is obtained from Eq. (A5).

B. Asymptotic orders in the contravariant spatial metric tensor

Using matrix notation, the contravariant metric tensor $\boldsymbol{g}^{ij}$ is computed from the covariant $\boldsymbol{g}_{ij}$ one using the relation $\boldsymbol{g}^{ij} = (\boldsymbol{g}_{ij})^{-1}$. For this aim, it is convenient to introduce the decomposition

$$
\boldsymbol{T} = \boldsymbol{T}_1 + \boldsymbol{T}_2 \; ; \qquad \text{(B8)}
$$

with

$$
\boldsymbol{T}_1 = \begin{bmatrix} a_{\alpha\beta} & 0 \\ 0 & 1 \end{bmatrix} \; ; \qquad \text{(B9)}
$$

and

$$
\boldsymbol{T}_2 = \begin{bmatrix} O(\delta) & O(\delta) \\ O(\delta) & 0 \end{bmatrix} \; ; \qquad \text{(B10)}
$$

where $\boldsymbol{T}^{-1} = \text{diag}\{a^{-1}_{\alpha\beta}, 1\}$ is regular if $a_{\alpha\beta}$ is regular too. Then, using Eq. (A5, B8),

$$
\begin{aligned}
\boldsymbol{g}^{ij} &= [g^{ij}] = \boldsymbol{S}^{-1} \boldsymbol{T}^{-1} \boldsymbol{S}^{-1} \\
&= \boldsymbol{S}^{-1} \boldsymbol{T}_1^{-1} \left(\boldsymbol{I} + \boldsymbol{T}_2 \boldsymbol{T}_1^{-1} \right)^{-1} \boldsymbol{S}^{-1} \\
&= \boldsymbol{S}^{-1} \boldsymbol{T}_1^{-1} \left(\boldsymbol{I} - \boldsymbol{T}_2 \boldsymbol{T}_1^{-1} + (\boldsymbol{T}_2 \boldsymbol{T}_1^{-1})^2 + ... \right)^{-1} \boldsymbol{S}^{-1} \\
&= \boldsymbol{S}^{-1} \left[\boldsymbol{T}_1^{-1} + O(\delta) \right] \boldsymbol{S}^{-1} \\
&= \text{diag}\{1, 1, \delta^{-1}\} \times \\
&\times \begin{bmatrix} a^{-1}_{\alpha\beta} + O(\delta) & O(\delta) \\ O(\delta) & 1 + O(\delta) \end{bmatrix} \text{diag}\{1, 1, \delta^{-1}\} \\
&= \begin{bmatrix} a^{-1}_{\alpha\beta} + O(\delta) & O(1) \\ O(1) & \delta^{-2} + O(\delta^{-1}) \end{bmatrix} \; ;
\end{aligned} \qquad \text{(B11)}
$$

where the last line in Eq. (B11) produces the second column of Table 1.

C. Asymptotic orders in the tangential component of the momentum equation

The asymptotic orders in the tangential component of the momentum equation are determined as follows. First, the convective acceleration is given by

$$
\begin{aligned}
u^i u^\gamma_{,i} &= u^\alpha u^\gamma_{,\alpha} + u^3 \frac{\partial u^\gamma}{\partial \eta^3} + u^3 u^h \begin{Bmatrix} \gamma \\ h\ 3 \end{Bmatrix} \\
&= u^\alpha u^\gamma_{,\alpha} + u^3 \frac{\partial u^\gamma}{\partial \eta^3} + O(\delta) \; ;
\end{aligned} \qquad \text{(C12)}
$$

Next, the reduced pressure gradient is split in normal and tangential components, and taken into account the second line of Eq. (D23),

$$\begin{aligned} g^{\alpha j}\frac{\partial p}{\partial \eta^j} &= g^{\alpha\beta}\frac{\partial p}{\partial \eta^\beta} + g^{\alpha 3}\frac{\partial p}{\partial \eta^3} \\ &= g^{\alpha\beta}\frac{\partial p}{\partial \eta^\beta} + O(1)O(\delta) \\ &= g^{\alpha\beta}\frac{\partial p}{\partial \eta^\beta} + O(\delta)\ . \end{aligned} \tag{C13}$$

Finally, for the viscous dissipation, the first covariant derivative of u^γ is given by

$$u^\gamma_{,i} = \frac{\partial u^\gamma}{\partial \eta^i} + \begin{Bmatrix} \gamma \\ h\ i \end{Bmatrix} u^h \quad . \tag{C14}$$

As in the $\{\gamma, h\ i\}$ symbol the special one $\{3, \beta\ \gamma\}$ is not present, all the velocities u^h are assumed of $O(1)$, and taking into account the asymptotic values in Table 1, it follows that

$$u^\gamma_{,i} = \frac{\partial u^\gamma}{\partial \eta^i} + O(\delta)\ ; \tag{C15}$$

and from Eq. (C15), when $i = 3$,

$$\frac{\partial u^\gamma_{,3}}{\partial \eta^3} = \frac{\partial^2 u^\gamma}{\partial \eta^3 \partial \eta^3} + O(\delta)\ . \tag{C16}$$

The second covariant derivative of the tangential velocity u^γ is

$$u^\gamma_{,ij} = \frac{\partial u^\gamma_{,i}}{\partial \eta^j} \begin{Bmatrix} \gamma \\ h\ j \end{Bmatrix} u^h_{,i} \begin{Bmatrix} h \\ i\ j \end{Bmatrix} u^\gamma_{,h} \quad ; \tag{C17}$$

in the first symbol on the r.h.s., the special case $\{3, \beta\gamma\}$ is not present either except for the second symbol. There are four cases to be considered:

1. when i, j run over $\{1, 2\}$,

$$\begin{aligned} u^\gamma_{,\alpha\beta} &= \frac{\partial u^\gamma_{,\alpha}}{\partial \eta^\beta} + \begin{Bmatrix} \gamma \\ h\ \beta \end{Bmatrix} u^h_{,\alpha} - \begin{Bmatrix} h \\ \alpha\ \beta \end{Bmatrix} u^\gamma_{,h} \\ &= \frac{\partial u^\gamma_{,\alpha}}{\partial \eta^\beta} + \begin{Bmatrix} \gamma \\ \lambda\ \beta \end{Bmatrix} u^\lambda_{,\alpha} + \begin{Bmatrix} \gamma \\ 3\ \beta \end{Bmatrix} u^3_{,\alpha} \\ &- \begin{Bmatrix} \lambda \\ \alpha\ \beta \end{Bmatrix} u^\gamma_{,\lambda} - \begin{Bmatrix} 3 \\ \alpha\ \beta \end{Bmatrix} u^\gamma_{,3} \\ &= O(1) + O(1) + O(\delta) + O(1) + O(\delta^{-1}) \\ &= O(\delta^{-1}) \quad \text{and} \quad g^{\alpha\beta} = O(1) \quad ; \end{aligned} \tag{C18}$$

2. when $i = 3$ and $j \neq 3$,

$$\begin{aligned} u^\gamma_{,3\beta} &= \frac{\partial u^\gamma_{,3}}{\partial \eta^\beta} + \begin{Bmatrix} \gamma \\ h\ \beta \end{Bmatrix} u^h_{,3} - \begin{Bmatrix} h \\ 3\ \beta \end{Bmatrix} u^\gamma_{,h} \\ &= \frac{\partial u^\gamma_{,3}}{\partial \eta^\beta} + \begin{Bmatrix} \gamma \\ \lambda\ \beta \end{Bmatrix} u^\lambda_{,3} + \begin{Bmatrix} \gamma \\ 3\ \beta \end{Bmatrix} u^3_{,3} \\ &- \begin{Bmatrix} \lambda \\ 3\ \beta \end{Bmatrix} u^\gamma_{,\lambda} - \begin{Bmatrix} 3 \\ 3\ \beta \end{Bmatrix} u^\gamma_{,3} \\ &= O(1) + O(1) + O(\delta) + O(\delta) + O(\delta) \\ &= O(1) \quad \text{and} \quad g^{3j} = O(1) \quad ; \end{aligned} \tag{C19}$$

3. the case $i \neq 3$ and $j = 3$ is analogous to the previous one;
4. finally when $i = j = 3$,

$$\begin{aligned} u^\gamma_{,33} &= \frac{\partial u^\gamma_{,3}}{\partial \eta^3} + \begin{Bmatrix} \gamma \\ h\ 3 \end{Bmatrix} u^h_{,3} - \begin{Bmatrix} h \\ 3\ 3 \end{Bmatrix} u^\gamma_{,h} \\ &= \frac{\partial u^\gamma_{,3}}{\partial \eta^3} + \begin{Bmatrix} \gamma \\ \lambda\ 3 \end{Bmatrix} u^\lambda_{,3} + \begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} u^3_{,3} \\ &- \begin{Bmatrix} \lambda \\ 3\ 3 \end{Bmatrix} u^\gamma_{,\lambda} - \begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} u^\gamma_{,3} \\ &= O(1) + O(\delta) + O(\delta^2)O(\delta^2) \\ &+ O(\delta^2) + O(\delta^2) \\ &= O(1) \quad \text{and} \quad g^{33} = O(\delta^{-2}) \quad . \end{aligned} \tag{C20}$$

Collecting Eqs. (C16, C18, C20) and using the standard assumption that the non-dimensional boundary layer thickness grows as $\delta = O(1/\sqrt{Re})$ which, in turn, is equivalent to $Re^{-1} = O(\delta^2)$, the dissipation of the tangential momentum is reduced to

$$\begin{aligned} Re^{-1} g^{ij} u^\gamma_{,ij} &= \frac{Re^{-1}}{\delta^2} u^\gamma_{,33} + Re^{-1} O(\delta^{-1}) \\ &= \frac{Re^{-1}}{\delta^2} \frac{\partial^2 u^\gamma}{\partial \eta^3 \partial \eta^3} + O(1)\ . \end{aligned} \tag{C21}$$

D. Asymptotic orders in the normal component of the momentum equation

The asymptotic orders in the normal component of the momentum equation are determined as follows:

1. The convective acceleration is given by

$$\begin{aligned} u^i u^3_{,i} &= u^i \frac{\partial u^3}{\partial \eta^i} + \begin{Bmatrix} 3 \\ h\ i \end{Bmatrix} u^h u^i \\ &= u^i \frac{\partial u^3}{\partial \eta^i} + \begin{Bmatrix} 3 \\ \alpha\ \beta \end{Bmatrix} u^\alpha u^\beta + \begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} u^3 u^3\ ; \\ &= O(1) + O(\delta^{-1}) + O(\delta^2) \\ &= O(\delta^{-1})\ ; \end{aligned} \tag{D22}$$

2. The reduced pressure gradient is decomposed as

$$\begin{aligned} g^{3j}\frac{\partial p}{\partial \eta^j} &= g^{33}\frac{\partial p}{\partial \eta^3} + g^{3\alpha}\frac{\partial p}{\partial \eta^\alpha} \\ &= \frac{1}{\delta^2}\frac{\partial p}{\partial \eta^3} + O(1)O(1) \\ &= O(\delta^{-1}) \quad ; \end{aligned} \tag{D23}$$

while $\quad \dfrac{\partial p}{\partial \eta^3} = O(\delta) \quad .$

3. For the viscous dissipation, the first covariant derivative of the normal velocity u^3 is given by

$$u^3_{,i} = \frac{\partial u^3}{\partial \eta^i} + \begin{Bmatrix} 3 \\ h\ i \end{Bmatrix} u^h\ ; \tag{D24}$$

which is split as

$$\begin{aligned} u^3_{,3} &= \frac{\partial u^3}{\partial \eta^3} + \begin{Bmatrix} 3 \\ h\ 3 \end{Bmatrix} u^h \\ &= O(1) + O(\delta) \\ &= O(1)\ ; \end{aligned} \tag{D25}$$

and

$$\begin{aligned} u^3_{,\alpha} &= \frac{\partial u^3}{\partial \eta^\alpha} + \begin{Bmatrix} 3 \\ \gamma\ \alpha \end{Bmatrix} u^\gamma + \begin{Bmatrix} 3 \\ 3\ \alpha \end{Bmatrix} u^3 \\ &= O(1) + O(\delta^{-1}) + O(\delta) \\ &= O(\delta^{-1})\ ; \end{aligned} \tag{D26}$$

and the first covariant derivative of the tangential velocity u^γ is given by

$$\begin{aligned} u^\gamma_{,3} &= \frac{\partial u^\gamma}{\partial \eta^3} + \begin{Bmatrix} \gamma \\ h\ 3 \end{Bmatrix} u^h \\ &= \frac{\partial u^\gamma}{\partial \eta^3} + \begin{Bmatrix} \gamma \\ \lambda\ 3 \end{Bmatrix} u^\lambda + \begin{Bmatrix} \gamma \\ 3\ 3 \end{Bmatrix} u^3 \\ &= O(1) + O(\delta) + O(\delta^2) \\ &= O(1)\ . \end{aligned} \tag{D27}$$

The second covariant derivative of the normal velocity u^3 is

$$u^3_{,ij} = \frac{\partial u^3_{,i}}{\partial \eta^j} + \begin{Bmatrix} 3 \\ h\ j \end{Bmatrix} u^h_{,i} - \begin{Bmatrix} h \\ i\ j \end{Bmatrix} u^3_{,h}\ . \tag{D28}$$

There are four cases to be considered:

i when i, j run over $\{1, 2\}$,

$$\begin{aligned} u^3_{,\alpha\beta} &= \frac{\partial u^3_{,\alpha}}{\partial \eta^\beta} + \begin{Bmatrix} 3 \\ h\ \beta \end{Bmatrix} u^h_{,\alpha} - \begin{Bmatrix} h \\ \alpha\ \beta \end{Bmatrix} u^3_{,h} \\ &= \frac{\partial u^3_{,\alpha}}{\partial \eta^\beta} + \begin{Bmatrix} 3 \\ \lambda\ \beta \end{Bmatrix} u^\lambda_{,\alpha} + \begin{Bmatrix} 3 \\ 3\ \beta \end{Bmatrix} u^3_{,\alpha} \\ &\quad - \begin{Bmatrix} \lambda \\ \alpha\ \beta \end{Bmatrix} u^3_{,\lambda} - \begin{Bmatrix} 3 \\ \alpha\ \beta \end{Bmatrix} u^3_{,3} \\ &= O(1) + O(\delta^{-1})O(1) \\ &\quad + O(\delta)O(\delta^{-1}) + O(1)O(\delta^{-1}) \\ &\quad + O(\delta^{-1})O(1) \\ &= O(\delta^{-1}) \text{ and } g^{\alpha\beta} = O(1)\ ; \end{aligned} \tag{D29}$$

ii when $i = 3$ and $j \neq 3$,

$$\begin{aligned} u^3_{,3\beta} &= \frac{\partial u^3_{,3}}{\partial \eta^\beta} + \begin{Bmatrix} 3 \\ h\ \beta \end{Bmatrix} u^h_{,3} - \begin{Bmatrix} h \\ 3\ \beta \end{Bmatrix} u^3_{,h} \\ &= \frac{\partial u^3_{,3}}{\partial \eta^\beta} + \begin{Bmatrix} 3 \\ \lambda\ \beta \end{Bmatrix} u^\lambda_{,3} + \begin{Bmatrix} 3 \\ 3\ \beta \end{Bmatrix} u^3_{,3} \\ &\quad - \begin{Bmatrix} \lambda \\ 3\ \beta \end{Bmatrix} u^3_{,\lambda} - \begin{Bmatrix} 3 \\ 3\ \beta \end{Bmatrix} u^3_{,3} \\ &= O(1) + O(\delta^{-1})O(1) \\ &\quad + O(\delta^2)O(1) \\ &\quad + O(\delta^2)O(1) + O(\delta^2)O(1) \\ &= O(\delta^{-1}) \text{ and } g^{3j} = O(1)\ ; \end{aligned} \tag{D30}$$

iii the case $i \neq 3$ and $j = 3$ is analogous to the previous one;

iv finally when $i = j = 3$,

$$\begin{aligned} u^3_{,33} &= \frac{\partial u^3_{,3}}{\partial \eta^3} + \begin{Bmatrix} 3 \\ h\ 3 \end{Bmatrix} u^h_{,3} - \begin{Bmatrix} h \\ 3\ 3 \end{Bmatrix} u^3_{,h} \\ &= \frac{\partial u^3_{,3}}{\partial \eta^3} + \begin{Bmatrix} 3 \\ \lambda\ 3 \end{Bmatrix} u^\lambda_{,3} + \begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} u^3_{,3} \\ &\quad - \begin{Bmatrix} \lambda \\ 3\ 3 \end{Bmatrix} u^3_{,\lambda} - \begin{Bmatrix} 3 \\ 3\ 3 \end{Bmatrix} u^3_{,3} \\ &= O(1) + O(\delta)O(1) \\ &\quad + O(\delta^2)O(1) \\ &\quad + O(\delta^2)O(1) + O(\delta^2)O(1) \\ &= O(1) \text{ and } g^{33} = O(\delta^{-2})\ . \end{aligned} \tag{D31}$$

Collecting Eqs. (D29, D30, D31) and using again the standard assumption $Re^{-1} = O(\delta^2)$, the dissipation of the normal momentum is reduced to

$$\begin{aligned} Re^{-1} g^{ij} u^3_{,ij} &= O(\delta^2)\ O(\delta^{-2})\ O(\delta^{-1}) \\ &= O(\delta^{-1})\ . \end{aligned} \tag{D32}$$

E. Asymptotic orders in the Christoffel symbols of first kind

The Christoffel symbols of first kind are calculated with

$$[ij, k] = \frac{1}{2} \left[\frac{\partial g_{ik}}{\partial \eta^j} + \frac{\partial g_{jk}}{\partial \eta^i} - \frac{\partial g_{ij}}{\partial \eta^k} \right]\ ; \tag{E33}$$

there are six types of these symbols depending on whether the affixes i, j, k are over the smooth surface (η^1 or η^2), or along the normal η^3. The order of these symbols with respect to the expansion parameter δ, with $0 < \delta \ll 1$, are estimated from the order of the metric tensor as given as

$$\begin{aligned} 2\ [\alpha\beta, \gamma] &= \frac{\partial g_{\alpha\gamma}}{\partial \eta^\beta} + \frac{\partial g_{\beta\gamma}}{\partial \eta^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial \eta^\gamma} \\ &= O(1) + O(1) + O(1) \\ &= O(1)\quad ; \end{aligned} \tag{E34}$$

$$\begin{aligned} 2\ [\alpha\beta, 3] &= \frac{\partial g_{\alpha 3}}{\partial \eta^\beta} + \frac{\partial g_{\beta 3}}{\partial \eta^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial \eta^3} \\ &= O(\delta^2) + O(\delta^2) \\ &\quad - \frac{\partial}{\partial \eta^3} \left[a_{\alpha\beta} + \eta^3 O(\delta) \right] \\ &= O(\delta)\quad ; \end{aligned} \tag{E35}$$

$$\begin{aligned} 2\ [\alpha 3, \beta] &= \frac{\partial g_{\alpha\beta}}{\partial \eta^3} + \frac{\partial g_{3\beta}}{\partial \eta^\alpha} - \frac{\partial g_{\alpha 3}}{\partial \eta^\beta} \\ &= O(\delta) + O(\delta^2) + O(\delta^2) = O(\delta)\quad ; \end{aligned} \tag{E36}$$

$$\begin{aligned} 2\ [3\beta, 3] &= \frac{\partial g_{33}}{\partial \eta^\beta} + \frac{\partial g_{\beta 3}}{\partial \eta^3} - \frac{\partial g_{3\beta}}{\partial \eta^3} = \frac{\partial g_{33}}{\partial \eta^\beta} \\ &= \frac{\partial \delta^2}{\partial \eta^\beta} = O(\delta^2)\quad ; \end{aligned} \tag{E37}$$

$$2\ [33,\gamma] = \frac{\partial g_{3\gamma}}{\partial \eta^3} + \frac{\partial g_{3\gamma}}{\partial \eta^3} - \frac{\partial g_{33}}{\partial \eta^\gamma} = O(\delta^2) \quad ; \tag{E38}$$

$$2\ [33,3] = \frac{\partial g_{33}}{\partial \eta^3} + \frac{\partial g_{33}}{\partial \eta^3} - \frac{\partial g_{33}}{\partial \eta^3} = \frac{\partial g_{33}}{\partial \eta^3} = 0 \quad . \tag{E39}$$

F. Asymptotic orders in the Christoffel symbols of second kind

The order of Christoffel symbols of second kind related to the expansion parameter δ, with $0 < \delta \ll 1$, is also determined from the order of the metric tensor, as follows

$$\begin{aligned}\{\alpha,\beta\gamma\} &= g^{\alpha\mu}[\beta\gamma,\mu] + g^{\alpha 3}[\beta\gamma,3] \\ &= O(1)\ O(1) + O(1)\ O(\delta) \\ &= O(1) \quad ;\end{aligned} \tag{F40}$$

$$\begin{aligned}\{3,\beta\gamma\} &= g^{3\mu}[\beta\gamma,\mu] + g^{33}[\beta\gamma,3] \\ &= O(1)\ O(1) + O(\delta^{-2})\ O(\delta) \\ &= O(\delta^{-1}) \quad ;\end{aligned} \tag{F41}$$

$$\begin{aligned}\{\beta,\alpha 3\} &= g^{\beta\mu}[\alpha 3,\mu] + g^{\beta 3}[\alpha 3,3] \\ &= O(1)\ O(\delta) + O(1)\ O(\delta^2) \\ &= O(\delta) \quad ;\end{aligned} \tag{F42}$$

$$\begin{aligned}\{3,\alpha 3\} &= g^{3\mu}[\alpha 3,\mu] + g^{33}[\alpha 3,3] \\ &= O(1)\ O(\delta) + O(\delta^{-2})\ O(\delta^2) \\ &= O(\delta) \quad ;\end{aligned} \tag{F43}$$

$$\begin{aligned}\{\alpha,33\} &= g^{\alpha\mu}[33,\mu] + g^{\alpha 3}[33,3] \\ &= O(1)\ O(\delta^2) + g^{\alpha 3} \\ &= O(\delta^2) \quad ;\end{aligned} \tag{F44}$$

$$\begin{aligned}\{3,33\} &= g^{3\mu}[33,\mu] + g^{33}[33,3] \\ &= O(1)\ O(\delta^2) + g^{33} \\ &= O(\delta^2) \quad .\end{aligned} \tag{F45}$$

Note that in Eqs. (F40) there is only one special case given by the symbol $\{3,\beta\ \gamma\} = O(\delta^{-1})$, while the remaining ones are $O(\delta)$ or $O(\delta^2)$, and that $\{\alpha, j\ k\} = O(\delta)$.

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